Effective Field Theory in the Three-Nucleon System

Michael H. Kesden

Princeton University, Class of 2000

mhkesden@princeton.edu

August 20, 1999

Abstract

At low energies, the coupling constant of the strong nuclear force becomes greater than unity. This prevents a perturbative analysis of scattering amplitudes in quantum chromodynamics. The techniques of effective field theory (EFT), in which the effects of high-energy physics are approximated by local interactions, provide a valuable new method of analyzing the two-nucleon and three-nucleon systems at these low energies. To demonstrate these new techniques, a toy potential model is chosen and its corresponding effective Lagrangian density is determined.

1 Introduction

The strong nuclear force, which is responsible for binding quarks into nucleons and nucleons into nuclei, is governed by the portion of the Standard Model known as quantum chromodynamics (QCD). The coupling constant α_s that determines the strength of interactions in QCD runs with the momentum q as [1]:

$$\alpha_s(|q^2|) = \frac{\alpha_s(\mu^2)}{1 + (\alpha_s(\mu^2)/12\pi)(11n - 2f)\ln(|q^2|/\mu^2)} \tag{1}$$

where n is the number of colors in the Standard Model, f is the number of flavors, and μ is the momentum scale at which the QCD coupling constant is normalized. Although asymptotic freedom guarantees that α_s becomes small at large q, for low momenta it is greater than unity. In this regime, Feynman diagrams involving more vertices will make a greater contribution to the amplitude for a given scattering process than lower order diagrams, preventing a perturbative expansion. In one technique for performing nonperturbative QCD calculations known as lattice gauge theory, quantum fields are evaluated only at the quantized points of a discrete lattice rather than in the continuum limit. Effective field theory offers an alternative approach, in which interactions below a certain length scale are approximated by a series of local operators and their derivatives, resulting in a power series expansion in momentum multiplied by this short distance scale.

2 Effective Field Theory

A final quantum field theory is entirely described by its Lagrangian density function, which specifies both the fields involved in the theory and the interactions between them. The theory is final in that it is valid at all energies and is not merely a low-energy approximation of some other final theory. In evaluating the Feynman diagrams that contribute to the amplitude of scattering processes beyond tree level, loops are encountered that involve integrating over intermediate states with momenta ranging from zero to infinity in magnitude. Final quantum field theories must be renormalizable in that they must accommodate these infinite momenta in a manner that results in finite scattering amplitudes, because these scattering amplitudes are directly related to physical observables. While renormalizability is an essential property of any final theory considered to be valid at all energy scales, it is not fundamental to an effective field theory describing scattering at external momenta k such that $k \ll \Lambda$, where Λ is a momentum cutoff delineating the regime at which the effective theory becomes invalid [2]. There is no physical justification for demanding renormalizability from a low-energy effective field theory, and in fact this requirement artificially constrains the low-energy behavior that might be produced by a perfectly renormalizable high-energy final theory.

One method of dealing with the divergences that arise in a nonrenormalizable effective field theory is the introduction of a momentum cutoff Λ which places an upper bound on the momenta of intermediate states that occur when evaluating loops in Feynman diagrams. Intermediate states with momenta $q > \Lambda$ do contribute to the low-energy behavior of the theory through Feynamn diagrams involving loops, and their removal must be accompanied by the introduction of new terms in the Lagrangian density that mimic their behavior at low energies. The incredible versatility and applicability of effective field theory derives from the fact that these new terms must be local and that only a finite number of these new terms are required to reproduce any behavior to finite order in the expansion parameter k/Λ. The locality of the new terms in the Lagrangian can be justified heuristically by simple quantum mechanical arguments [2]. The intermediate states with momenta $q > \Lambda$ that have been removed have energies much greater than that of the real external states with momenta k, and as such must be highly virtual. The uncertainty relation that exists between energy and time implies that these intermediate states can only exist for a short duration, during which time they can only propogate a distance of order $1/\Lambda$. This distance is small compared to the wavelength of the external states which is of order $1/k$, suggesting that the effect of the intermediate high momenta states can be accurately reproduced by local interactions in the Lagrangian. These local interactions appear as delta functions and their derivatives in position space, and when Fourier transformed lead to coupling constants that are even polynomials in the expansion parameter k/Λ. The second important result of effective field theory, that to finite order in k/Λ only a finite number of new terms must be introduced into the Lagrangian density to recreate any behavior, can be seen from the fact that only a finite number of acceptable terms are possible. The fact that these terms must be local places a significant constraint on the forms they can take, as does the fact that they must preserve the underlying symmetries of the final quantum theory, which in the case of nuclear theory involve among others gauge symmetry, chiral symmetry, and Galilean invariance [3]. There are only several different fields in the quantum theory, and terms of higher order in these fields are suppressed by powers of Λ, the only scale present in the theory in the natural case. Once the finite coupling constants from these few new terms are determined from experiment, the effective theory can describe all behavior to any finite order in the expansion parameter k/Λ.

3 Toy Model

The utility of an analysis using effective field theory can be readily demonstrated by a comparison by comparing the results it provides to explicit calculations performed in the context of a toy model. Scattering in nonrelativistic quantum mechanics is governed by the Lippmann-Schwinger equation, which in momentum space representation is given by [4]:

$$< k|t(E)|k' >=< k|V|k' > +\frac{1}{(2\pi)^3}\int \frac{< k|V|p >< p|t(E)|k' >}{E-p^2}dp \tag{2}$$

Using a partial wave decomposition, the potential can be expressed as a summation of spherical harmonics [4]:

$$< k|V|k' >= \sum_{l,m} V_l(k,k')Y_{lm}(\hat{k})Y^*_{lm}(\hat{k}') \tag{3}$$

A similar decomposition can be performed on the T-matrix, allowing the Lippmann-Schwinger equation (2) to be written separately for each partial wave l [4]:

$$t_l(k,k',E) = V_l(k,k') + \frac{1}{(2\pi)^3}\int \frac{V_l(k,p)t_l(p,k',E)}{E-p^2}p^2dp \tag{4}$$

Solving this integral equation is computationally intensive for arbitrary potentials, but can be solved algebraically for separable potentials of the form [4]:

$$V_l(k,k') = \lambda_l V_l(k)V_l(k') \tag{5}$$

An integral equation related to the Lippmann-Schwinger equation arises during the analysis of three-nucleon scattering in nonrelativistic quantum field theory, and for this reason we choose for our toy model the separable Yamaguchi potential:

$$V_l(k) = \frac{\sqrt{\Delta\beta}}{\alpha^2 + k^2} \tag{6}$$

where α and β are adjustable parameters and Δ is a scaling parameter that cancels during the calculation of any observable. A potential that is separable in momentum space will have a Fourier transform that is separable in position space as well. The Yamaguchi potential in position representation is given by:

$$V_l(r, r') = \frac{\Delta\beta}{(4\pi)^2} \frac{e^{-\alpha(r+r')}}{rr'} \tag{7}$$

While this potential is not local, the decaying exponential implies that it is sufficiently short-ranged as to be ultraviolet-convergent and thus renormalizable. It is therefore a valid final quantum theory for all energy regimes. Having chosen this toy model, we proceed to calculate scattering lengths for two and three-nucleon scattering.

4 Two-nucleon Scattering

The general Lagrangian density for particle/bound-state scattering is given by [5]:

$$L = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi + \Delta T^\dagger T - f T^\dagger \psi\psi - f^* T \psi^\dagger \psi^\dagger \tag{8}$$

where ψ and T are fields representing the particle and bound-state respectively and f is a momentum dependent vertex factor given in our toy model by the Yamaguchi potential (6). The momentum at which this vertex factor is evaluated for a Feynman diagram with particular kinematics is illustrated in Figure 1. Elastic nuclear scattering is norm-preserving, so the only effect of the scattering can be a shift in the phase of each outgoing partial wave. The amplitude of the scattering process can be determined by summing those Feynman diagrams which contribute to that process. A first step in this process is evaluating the loop integral that appears in Feynman diagrams such as that depicted in Figure 2. These kinematics correspond to the loop integral:

$$L = \int \frac{d^4q}{(2\pi)^4} \frac{i}{\frac{k^2}{2m} + q_0 - \frac{q^2}{2m} + i\epsilon} \frac{i}{\frac{k^2}{2m} - q_0 - \frac{q^2}{2m} + i\epsilon} (-if^*(q))(-if(q))\Delta \tag{9}$$

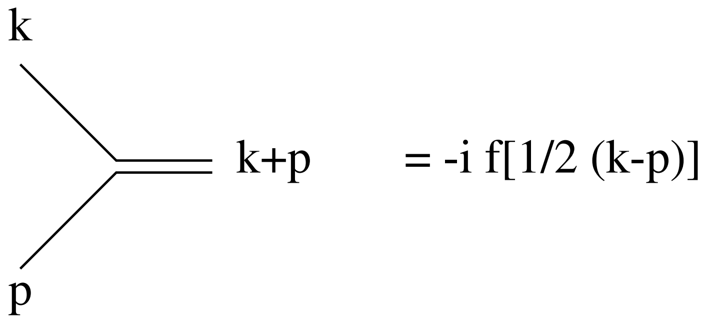

Figure 1: Momentum at which vertex factors appearing in the Lagrangian (8) are evaluated.

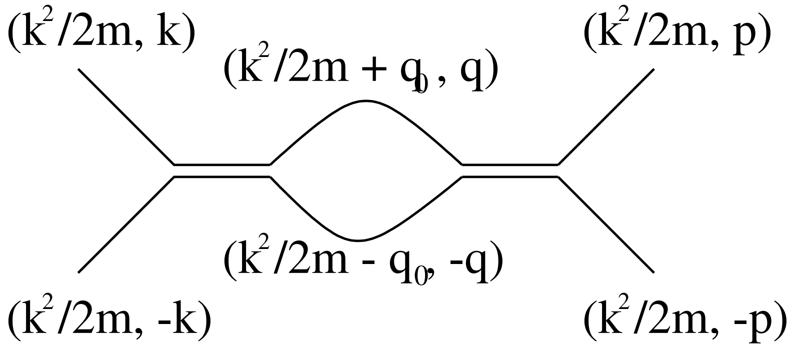

Figure 2: One-loop Feynman diagram contributing to two-nucleon scattering. The energy-momentum four-vector of the initial, intermediate, and final states are shown in the center-of-mass reference frame.

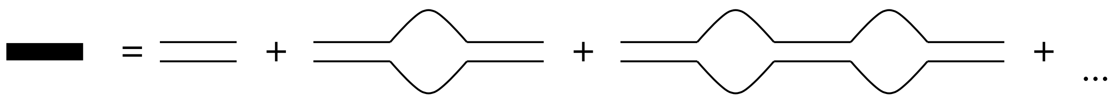

Figure 3: Bubble chain of Feynman diagrams that contribute to the dressed dibaryon propogator responsible for two-nucleon scattering.

where the first two factors in the integrand represent the upper and lower particle propogators, and the second two factors account for the left and right vertex factors. This integral is ultraviolet convergent for our Yamaguchi potential (6), and as such it is unnecessary to introduce a momentum cutoff. The result, after proper pole prescriptions are taken into account, is given by:

$$L = \frac{im\Delta\beta}{8\pi}\left(\frac{2ik+\alpha-\frac{k^2}{\alpha}}{(k^2+\alpha^2)^2}\right) \tag{10}$$

Having evaluated the loop integral L, the amplitude for two-nucleon scattering can be determined by summing the appropriate bubble chain of Feynman diagrams illustrated in Figure 3. This infinite summation results in the geometric series:

$$P = \sum_{l=0}^{\infty}\left(\frac{i}{\Delta}\right)2^l L^l \tag{11}$$

where the index l represents the number of loops, $\frac{i}{\Delta}$ is the bare dibaryon propogator, 2^l is a symmetry factor, and L is the loop integral (10). This geometric series can be summed like any convergent geometric series, resulting in the dressed dibaryon propogator:

$$P = \frac{i/\Delta}{1+\frac{m\beta}{4\pi}\left(\frac{2ik+\alpha-\frac{k^2}{\alpha}}{(k^2+\alpha^2)^2}\right)} \tag{12}$$

Proceeding from the dressed dibaryon propogator to the two-nucleon scattering amplitude simply consists of multiplying by the appropriate external vertices and symmetry factor:

$$iA = \left(\frac{-2\beta\Delta}{(k^2+\alpha^2)^2}\right)P = i\left(\frac{4\pi}{m}\right)\frac{1}{-\frac{2\pi}{m\beta}(k^2+\alpha^2)^2-\frac{\alpha}{2}+\frac{k^2}{2\alpha}-ik} \tag{13}$$

Nonrelativistic quantum mechanics indicates that the T-matrix elements $t_l(p,k,E)$ for each partial wave can be related to partial-wave amplitudes $f_l(k)$ defined by [6]:

$$f_l(k) \equiv -\frac{\pi t_l(p,k,E)}{k} \tag{14}$$

Each partial-wave amplitude is directly related to the amplitude determined from the Feynman diagrams, and can be defined in terms of a phase shift angle since elastic scattering is norm-preserving. This relation:

$$iA = i\left(\frac{4\pi}{m}\right) f_l(k) = i\left(\frac{4\pi}{m}\right)\frac{1}{k\cot\delta_l - ik} \tag{15}$$

allows the phase shifts for the Yamaguchi potential to be determined.

$$k\cot\delta_l = -\frac{2\pi\alpha^4}{m\beta} - \frac{\alpha}{2} + \left(\frac{1}{2\alpha} - \frac{4\pi\alpha^2}{m\beta}\right)k^2 - \frac{2\pi}{m\beta}k^4 \tag{16}$$

This result concurs with the effective range expansion, which dictates that for small momenta $k \ll \Lambda$ the phase shifts can be expanded in an even polynomial in momenta given by [3]:

$$k\cot\delta_l = -\frac{1}{a} + \frac{1}{2}\Lambda^2\sum_{n=0}^{\infty} r_n\left(\frac{k^2}{\Lambda^2}\right)^{n+1} \tag{17}$$

where a is the scattering length and r_0 is the effective range. It can be compared to the effective range expansion of the phase shifts predicted by an effective field theory, to determine the extent to which local interactions can recreate the behavior of the complete theory for momenta below that of the EFT's cutoff.

5 Nucleon/Bound-State Scattering

Having calculated the two-nucleon scattering amplitude, we can to proceed to a calculation of the three-nucleon scattering amplitude in the presence of a two nucleon bound state. As in the previous case, an infinite number of Feynman diagrams contribute to the scattering amplitude, and these diagrams must be summed in a consistent manner. The series of "pinball" diagrams contributing to nucleon/bound-state scattering as depicted in Figure 4 [5]. Unlike two-nucleon scattering, the infinite series of Feynman diagrams describing nucleon/bound-state scattering cannot be summed geometrically. A recursive relation between the diagrams can be established, resulting in an integral equation for the scattering amplitude t. The kinematics that will be used to evaluate this integral equation are described in Figure 5 [5]. The first diagram on the right-hand side of Figure 5 contributes an inhomogenous term to the integral equation:

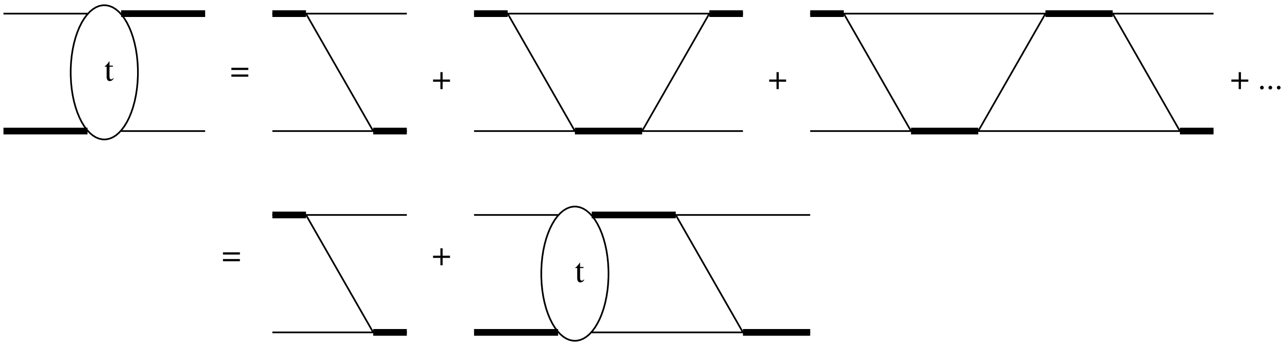

Figure 4: Pinball diagrams contributing to nucleon/bound-state scattering. The infinite series is equivalent to a reursive relation between diagrams.

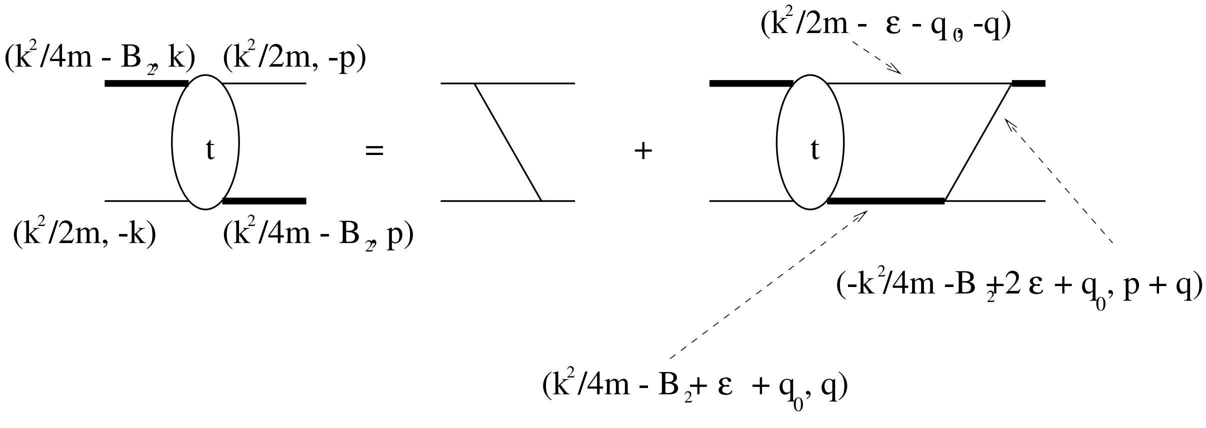

Figure 5: Kinematics for calculation of the nucleon/bound-state scattering amplitude in the center-of-mass system.

$$iA_{inh} = [-if^*(\sqrt{p^2 + p \cdot k + \frac{k^2}{4}})\sqrt{\Delta}]\frac{i}{-\frac{k^2}{4m} - B_2 + \varepsilon - \frac{1}{2m}(k^2 + 2p \cdot k + p^2) + i\epsilon} \tag{18}$$
$$[-if(\sqrt{k^2 + p \cdot k + \frac{p^2}{4}})\sqrt{\Delta}]$$

where B_2 is the binding energy of the dibaryon. The first and third factors in this expression represent the left and right external vertex factors respectively, while the second factor is a nucleon propogator. The second diagram in Figure 5 contributes the homogeneous term:

$$iA_{hom} = \int \frac{d^4q}{(2\pi)^4} it(k, q, \varepsilon + q_0) \frac{i}{\frac{k^2}{2m} - \varepsilon - q_0 - \frac{q^2}{2m} + i\epsilon} \frac{i}{-\frac{k^2}{4m} - B_2 + 2\varepsilon + q_0 - \frac{1}{2m}(q^2 + 2p \cdot q + p^2) + i\epsilon} \tag{19}$$
$$\frac{i}{\Delta} \frac{1}{1 + \frac{m}{\pi}\int dq \frac{q^2|f(q)|^2}{a^2+q^2}} [-if^*(\sqrt{p^2 + p \cdot q + \frac{q^2}{4}})\sqrt{\Delta}][-if(\sqrt{k^2 + p \cdot k + \frac{p^2}{4}})\sqrt{\Delta}]$$

where $-a^2 = mE - \frac{3q^2}{4} + i\epsilon$. For the Yamaguchi potential (6) of our toy model, these terms can be evaluated and partially simplified by projection onto the S-wave $l = 0$. The resulting one-dimensional integral equation has inhomogeneous and homogeneous components given by:

$$iA_{inh} = \frac{\Delta m \beta}{4\pi XYZ}[X \ln\left(\frac{\alpha^2 + p^2 + \frac{1}{4}k^2 - pk}{\alpha^2 + p^2 + \frac{1}{4}k^2 + pk}\right) + Y \ln\left(\frac{mB_2 + p^2 + \frac{1}{4}k^2 - pk}{mB_2 + p^2 + \frac{1}{4}k^2 + pk}\right) \tag{20}$$
$$+ Z \ln\left(\frac{\alpha^2 + k^2 + \frac{1}{4}p^2 - pk}{\alpha^2 + k^2 + \frac{1}{4}p^2 + pk}\right)]$$

where $X = \alpha^2 - \frac{3}{4}(p^2 - k^2) - mB_2$, $Y = \frac{3}{4}(p^2 - k^2)$, and $Z = mB_2 - \alpha^2$ and:

$$iA_{hom} = \frac{\alpha^2}{2\pi\beta}\int \frac{dq q^2 t(k,q)(\gamma + \alpha)^2}{X'Y'Z'}[X' \ln\left(\frac{\alpha^2 + p^2 + \frac{1}{4}q^2 - pq}{\alpha^2 + p^2 + \frac{1}{4}q^2 + pq}\right) + Y' \ln\left(\frac{-mE + p^2 + q^2 - pq}{-mE + p^2 + q^2 + pq}\right) \tag{21}$$
$$+ Z' \ln\left(\frac{\alpha^2 + q^2 + \frac{1}{4}p^2 - pq}{\alpha^2 + q^2 + \frac{1}{4}p^2 + pq}\right)]$$

where $X' = \alpha^2 - \frac{3}{4}p^2 + mE$, $Y' = \frac{3}{4}(p^q - q^2)$, $Z' = \gamma^2 - \alpha^2$, and $\gamma = \sqrt{\frac{3q^2}{4} - mE}$. This equation, while untenable analytically, is amenable to numerical computation which will provide the scattering amplitude as a function of initial and final momenta.

6 Effective Theory Approximation

In an effective field theory Lagrangian, interactions are local implying that coupling constants are sucessive derivatives of delta functions in position representation and even polynomials in momentum representation. Terms of this form are nonrenormalizable, but ultraviolet divergences can be avoided by introducing a momentum cutoff Λ. To simulate the two-nucleon and nucleon/bound-state scattering of our toy model to leading order, an effective Lagrangian of the form [5]:

$$L = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}}(T^\dagger \psi\psi + T\psi^\dagger\psi^\dagger) + hT^\dagger T\psi^\dagger\psi \tag{22}$$

is adopted, where unlike in the case of a complete quantum field theory the form factors are momentum independent. As discussed in Section 4, the first step in constructing the dressed dibaryon propogator is the evaluation of the loop integral depicted in Figure 1, however in this instance the integral is only evaluated from zero to the momentum cutoff Λ. The result of this integral is:

$$L = -g^2 \left(\frac{-im\Lambda}{2\pi^2} + \frac{mk}{2\pi} \right) \tag{23}$$

which when summed in a geometric series corresponding to the Feynman diagrams of Figure 3 leads to the two-nucleon scattering amplitude:

$$iA = \sum_{l=0}^{\infty} \left(\frac{i}{\Delta}^{l+1} \right) \left(\frac{-ig}{\sqrt{2}}^{2l+2} 2^{l+2} \right) L^l = i \left(\frac{4\pi}{m} \right) \frac{1}{-\frac{2\pi\Delta}{mg^2} - \frac{\Lambda}{\pi} - ik} \tag{24}$$

Initially this scattering amplitude appears suspect because it seem to imply that the phase shift, a physical observable, depends upon the momentum cutoff, an artificially chosen parameter. This dependency can be removed by introducing terms into the effective Lagrangian giving the mass and coupling constants a dependence on the momentum cutoff Λ as well. Once the appropriate coupling constant, mass, and wave function renormalizations have been performed, all observables in the effective theory will be cutoff independent. A calculation of the nucleon/bound-state scattering amplitude similar to that performed in Section 5 could be performed as well, but this time a three-nucleon force must be included as well corresponding to the last term in the effective Lagrangian (22). The incorporation of this three-nucleon interaction is in accordance with the principle discussed in Section 2 that all local interactions consistent with the symmetries in the underlying complete field theory must be included in the EFT. By adjusting the coupling constants present in the effective Lagrangian (22) to match the terms in the effective range expansion (16) of the toy model, all low-energy behavior can be recreated to leading order in the expansion parameter k/Λ. Higher order corrections can be produced by including local interactions in the effective Lagrangian with coupling constants that are delta function derivatives in position representation or even polynomials in the momentum representation.

7 Conclusion

Although the Yamamguchi potential introduced as our toy model was specifically chosen to produce simple analytic results, the techniques of effective field theory are applicable to more complicated potentials as well as direct experimental data. An effective Lagrangian containing only local interactions, once it has been properly regulated by either a momentum cutoff, dimensional regularization, or some other method, can reproduce the low-energy behavior of any underlying complete field theory. In some sense it is unrealistic to assume that an ultimate complete field theory will every be obtained, because it is unlikely that experiments will ever probe physics at the highest energy scales. In the meantime, effective field theory can serve as a valuable tool for describing and predicting physics in experimentally accessible regimes.

8 Acknowledgements

First and foremost, I would like to thank my advisor Dr. Paulo Bedaque for his near infinite wisdom and patience in introducing me to the principles of effective field theory. Also invaluable was the assistance of Professor Martin Savage, Dr. Daniel Phillips, Dr. Harald Greishammer, Gautam Rupak, and other members of the Effective Field Theory Group. Finally I would like to thank my hosts for the summer, Professor Wick Haxton, the Institute for Nuclear Theory, and the University of Washington. This work was supported by a grant from the National Science Foundation though the Research Experience for Undergraduates

Program.

References

[1] Griffiths, David. *Introduction to Elementary Particles.* New York: John Wiley & Sons, Inc., 1987.

[2] Lepage, G. Peter. "What is Renormalization?" Invited lectures given at the TASI-89 Summer School, Boulder, Colorado, June 4-30, 1989. Published in Boulder ASI 1989: 483-508.

[3] Kaplan, David B., Savage, Martin J., and Wise, Mark B. "Two-nucleon systems from effective field theory." Nuclear Physics B 534 (1998) 329-355.

[4] Amado, Ralph D. "Separable Interactions." *The Three-Body Problem.* Ed. W. K. Cheng. Lectures given at Brandeis University, Waltham, Massachusetts, 1967: 34-60.

[5] Bedaque, P. F., Hammer, H.-W., and van Kolck, U. "The Three-Boson System with Short-Range Interactions." Nuclear Physics A 646 (1999) 444-466, nucl-th/9811046.

[6] Sakurai, J. J. *Modern Quantum Mechanics.* New York: Addison-Wesley Publishing Company, Inc., 1994.