PHYS 560: Lectures During Week 5

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Abstract

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1 Gross Features of Binding Energies

Up to this point of the course we have been examining in detail the two-body interaction between nucleons. The hope has been that we could build up nuclei from by understanding pair-wise interactions. We have not given up on this hope and we will attempt to implement this approach at all stages of our analysis. However, we must introduce and analyse other important features of nuclei that have been known for many years in order to get a better idea of what the "big-picture" is.

One of the simplest measured quantities we can determine for a nucleus is its binding energy, $B(A, Z)$ which can depend on its global quantum numbers, electric charge and total nucleon number of the nucleus (baryon number). This quantity is defined in terms of the mass of the atom $M(A, Z)$, the mass of the neutron M_n and the mass of a hydrogen atom M_H. By definition

$$B(A, Z) = ZM_H + (A - Z)M_n - M(A, Z) \quad , \tag{1}$$

which is a positive quantity for bound states.

A quantity that is directly related to $B(A, Z)$ that is found to be essentially independent of A is the binding energy per nucleon B/A. The experimentally observed binding energy per nucleon is shown in the fig (1). One sees that over the range of nuclei that are stable (which we will get to later) the binding energy per nucleon is essentially constant, lying between 7.5 and about 8.8 MeV.

This behaviour tells us a lot about the nuclear force. Imagine that the interaction between nucleons was entirely yukawa type mediated by pion exchange, that dominates the long-distance part. For distances inside of an inverse pion mass, the interaction looks coulombic. We know that for a coulomb system the particles want to sit on top of each other, this is the configuration that has the lowest potential energy. The actual configuration is a trade off between the kinetic and potential, that is why there is a finite radius of the hydrogen atom. However, when we put together more and more particle that interact via coulomb type interactions, which are infinite range, then the interaction energy depends on the number of pair-wise interactions that exist. In the case of nuclear interactions this would correspond to a term dependent upon A^2. Clearly, such dependence is inconsistent with the nearly constant B/A. One can understand this behaviour if in fact a given nucleon interacts with only a few of the other nucleons in the nucleus, such that the number of pair-wise interactions is not growing with A^2. A nucleon

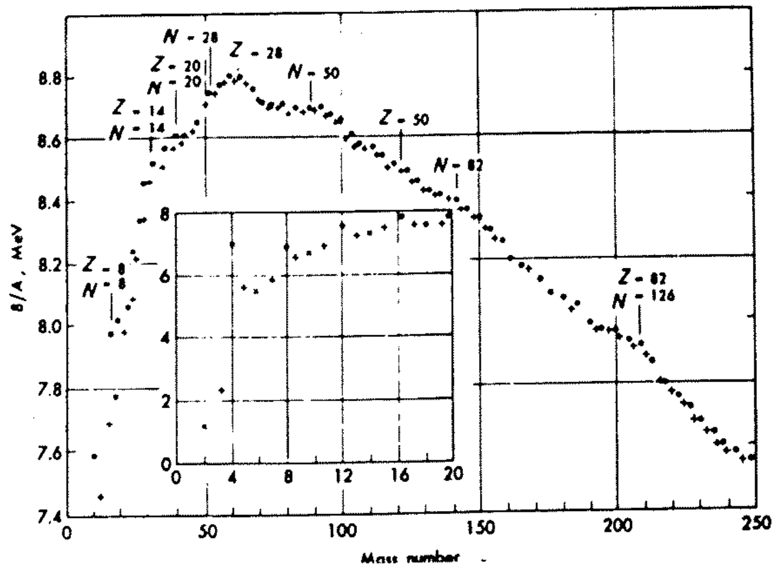

Figure 1: *Binding energy per nucleon*

on one side of the nucleus is far outside its interaction range of a nucleon on the other side of the nucleus. This picture also leads to the idea that the density of nuclear matter is constant. This is indeed true and we will return to this later.

2 Semi-Empirical Mass Formula for Nuclei

Lets us use the above general ideas to arrive at a mass formula that describes the gross behaviour of the binding energy of nuclei. Our argument will be at the classical level (mainly) and we consider nuclei to be nothing more than "blobs" or "chunks" of nuclear matter, without looking at the precise nature of the nuclear interaction.

Imagine that we did have nuclear interactions that saturates (a strong repulsive core to the nucleon-nucleon interaction that keeps the nucleons apart) then we would expect that the binding energy of a nucleus would depend on the volume of the nucleus. The volume of the nucleus is linear in the atomic number due to the saturation, and hence non-zero distance between nucleons, independent of the number of other nucleons

$$B_{\rm vol} \sim A \quad . \tag{2}$$

In the above argument we have neglected the fact that for nucleons near or at the surface of the nucleus, the number of nucleons they see (on average) is less than if they were in the nuclear medium itself. This means that we have to remove a contribution to the binding energy that depends on the area of the nuclear surface. Hence, we have a contribution of the form

$$B_{\rm surf.} \sim -A^{2/3} \quad , \tag{3}$$

as for an object of radius R, the volume is proportional to R^3, while the surface is proportional to R^2.

Let us pause here and take note of the form that these two terms give for the binding energy of a chunk of nuclear material. At this stage alone we have that

$$\begin{aligned} B &= \alpha\ A - \beta\ A^{\frac{2}{3}} \\ B/A &= \alpha\ -\ \beta \frac{1}{A^{\frac{1}{3}}} \quad , \end{aligned} \tag{4}$$

where α and β are constants that need to be determined from data. We see that this is a monotonically increasing function (forget the $A \sim 0$ bad behaviour as a single nucleon doesn't count as a nucleus). As we increase A, the function goes to a constant as the surface relative to the volume becomes vanishingly small. In fact, extrapolation of the data, we find that $\alpha = 15.8$ MeV. However, we see that this does not reproduce the large A trend of nuclei, which tends to be falling in the region of masses available to us.

A contribution that we have not considered is from the electromagnetic interaction. We have considered only the effects of nuclear interactions thus far, as we have assumed that nuclear forces are much stronger than electromagnetic forces. This statement is absolutely true, but the electromagnetic contributions have a compensating feature that makes them an important. The electromagnetic interaction is long range, and acts over the entire volume of the nucleus, which we know is in contrast to the nuclear interaction which is limited to essentially those nucleons within (at most) an inverse pion mass. In particular, if we have a nucleus of charge Z with $Z >> 1$ then the coulombic contribution is order Z^2, which is a big number.

You will recall that the potential energy of a charged sphere of radius R is $3Z^2/(5R)$ (this form arises from bringing infinitesimally thin shells from infinity, computing the work done, and then bringing more shells up until the charged sphere is formed of radius R). Writing this in terms of A we find a contribution of the form

$$B_{\rm coul} \sim \frac{Z^2}{A^{\frac{1}{3}}} \quad . \tag{5}$$

If we consider the case where $A = 2Z$ for arguments sake, then we find that the coulomb terms becomes relatively more important as we go to larger A. The repulsion of the protons in the medium lowers the binding energy from the naive volume and surface terms.

Lets look at the features of the binding energy per nucleon with just these three terms, all of which we have arrived at classically,

$$\begin{aligned} B &= \alpha\ A - \beta\ A^{\frac{2}{3}} - \epsilon\frac{Z^2}{A^{\frac{1}{3}}} \\ B/A &= \alpha\ -\ \beta\frac{1}{A^{\frac{1}{3}}}\ -\ \epsilon\frac{Z^2}{A^{\frac{4}{3}}} \quad . \end{aligned} \tag{6}$$

Let us set $Z = A/2$ to arrive at $B/A = \alpha\ -\ \beta\frac{1}{A^{\frac{1}{3}}}\ -\ \frac{\epsilon}{4}A^{\frac{2}{3}}$. In this form, it is clear that the coulomb terms is growing while the surface term is shrinking with increasing A. The is a stationary point of this function for a given A, we find by setting the partial derivative wrt A to vanish,

$$\begin{aligned} \frac{\partial}{\partial A} B/A &= \frac{\beta}{3A^{\frac{4}{3}}} - \frac{\epsilon}{6A^{\frac{1}{3}}} = 0 \\ A_0 &= \frac{2\beta}{\epsilon} \quad , \end{aligned} \tag{7}$$

from which we see that there is a "most stable" nucleus, one with the maximum binding energy per nucleon.

Before we get too bogged down at this point, lets move on, as we know there are other effects that have to be included in our semi-empirical mass formula, before we can rest and be happy. Let us think about we we know about cramming non-interacting fermions (we will justify this later) into a small volume. In a closed box, the fermions fill up the levels up to the fermi-level (by definition). There is one fermi level for the neutrons and one for the protons. Lets now consider the implications of this.

The number of states for a given species with g internal states is

$$N = g \int d^3k \quad , \tag{8}$$

and for our purposes we will lump the angular integral and degeneracy factor into a constant α, following the section in *Preston+Bhaduri, page 200.* The fermi levels for the neutrons k_F^n and protons k_F^p define another fermi-level k_F (the average fermi-level)by

$$\begin{aligned}
\frac{1}{2}A &= \alpha \int_0^{k_F} dk\ k^2 \\
\alpha &= \frac{3A}{2k_F^3} \\
Z &= \alpha \int_0^{k_F^p} dk\ k^2 \ = \ \frac{A}{2}\left(\frac{k_F^p}{k_F}\right)^3 \\
N &= \alpha \int_0^{k_F^n} dk\ k^2 \ = \ \frac{A}{2}\left(\frac{k_F^n}{k_F}\right)^3 \quad .
\end{aligned} \tag{9}$$

By forming ratios of the quantities in eq. (9) it is easy to show that, in terms of the quantities $I = N - Z$ (asymmetry) and $A = N + Z$ (mass number)

$$\begin{aligned}
k_F^n &= k_F \left(1 + \frac{I}{A}\right)^{\frac{1}{3}} \\
k_F^p &= k_F \left(1 - \frac{I}{A}\right)^{\frac{1}{3}} \quad ,
\end{aligned} \tag{10}$$

and hence we can define the difference between the individual and average fermi-momenta (expanding in the small parameter I/A)

$$\begin{aligned}
\delta^n &= k_F^n - k_F = k_F \left(\frac{I}{3A} - \frac{I^2}{9A^2}\right) \\
\delta^p &= k_F^p - k_F = k_F \left(-\frac{I}{3A} - \frac{I^2}{9A^2}\right) \quad .
\end{aligned} \tag{11}$$

We are really interested in the total energy of this non-interacting fermi gas, of which there is only kinetic energy, given by an integral over the kinetic energy operator $t(k)$ (which we will take to be the same for neutrons and protons, by isospin symmetry)

$$T = \alpha \int_0^{k_F^n} dk\ k^2\ t(k) \ + \ \alpha \int_0^{k_F^p} dk\ k^2\ t(k) \quad . \tag{12}$$

In order to compute what we are after, lets determine one of these integrals, and put it in the desirable form. We have (for a smoothly varying function $t(k)$)

$$\begin{aligned}
&\alpha \int_0^{k_F^n} \ dk\ k^2\ t(k) = \alpha \int_0^{k_F+\delta^n} \ dk\ k^2\ t(k) \\
&= \alpha \int_0^{k_F} \ dk\ k^2\ t(k)\ +\ \alpha \int_{k_F}^{k_F+\delta^n} \ dk\ k^2\ t(k) \\
&= \alpha \int_0^{k_F} \ dk\ k^2\ t(k)\ +\ \alpha \int_{k_F}^{k_F+\delta^n} \ dk\ k^2\ \left[t(k_F)\ +\ (k-k_F)\frac{\partial t}{\partial k}\ +\ ...\right] \\
&= \alpha \int_0^{k_F} \ dk\ k^2\ t(k)\ +\ \alpha\ \delta^n\ k_F^2\ t(k_F)\ +\ ... \qquad (13)
\end{aligned}$$

This is nothing more than a taylor series expansion about the average fermi-level k_F. We can write the energy of this fermi gas as the sum of the energy for $Z = N = A/2$ plus terms corresponding to $Z \neq N \neq A/2$. Deviations from the symmetry point $Z = N = A/2$ will be reflected in the binding energy formula, depending on δ^n and δ^p. We write $T = T_0 + T_{\rm sym} + ...$ and find that

$$\begin{aligned}
T_{\rm sym} &= \alpha k_F^2 t(k_F)\ (\delta^n + \delta^p) \\
&= \frac{3A}{2k_F^3} k_F^3\ t(k_F)\ \left(\frac{2I^2}{9A^2}\right) \\
&= \frac{1}{3}\ t(k_F)\ \frac{I^2}{A} \quad . \qquad (14)
\end{aligned}$$

Therefore, at lowest order in the asymmetry we expect to find a term proportional to $(N-Z)^2/A$ in our binding energy relation. So lets put it in!!

Finally, there is one more ingredient to our binding energy recipe. From our somewhat superficial look at nucleon-nucleon interactions we saw that the interaction depended upon what spin-isospin channel was scattering. This tells us that the binding energy of 2 protons, is the same as two neutrons, but different for a neutron-proton pair. So we define a discrete operator $\delta(A,Z)$ that $= 0$ for odd A nuclei, $= +1$ for odd-odd nuclei and $= -1$ for even-even nuclei.

Combining all these contributions to the binding energy, we find that

$$B(A,Z) = \alpha A - \beta A^{2/3} - \gamma\frac{(A-2Z)^2}{A} - \epsilon\frac{Z^2}{A^{\frac{1}{3}}} - \chi\frac{(A-2Z)^2}{A^{\frac{4}{3}}} - \eta\delta(A,Z) \qquad (15)$$

where the χ term that I have slipped in here is a surface asymmetry term. Fitting the observed masses leads to a parameter set

$$\begin{aligned}
\alpha &= 15.8\mathrm{MeV} \\
\beta &= 17.8\mathrm{MeV} \\
\gamma &= 23.75\mathrm{MeV} \\
\epsilon &= 0.71\mathrm{MeV} \\
\chi &= 33.22\mathrm{MeV} \\
\eta &= 12/\sqrt{A}\mathrm{MeV} \quad .
\end{aligned} \tag{16}$$

One sees that it is possible to trade off volume energy for asymmetry energy, i.e. additional neutrons at no additional cost or coulomb interactions. We can determine the value of Z that maximizes the binding energy for a given value of A, by simply taking a partial derivative. I have shown the result of that in the figure, and also a plot of B/A on this maximal trajectory.

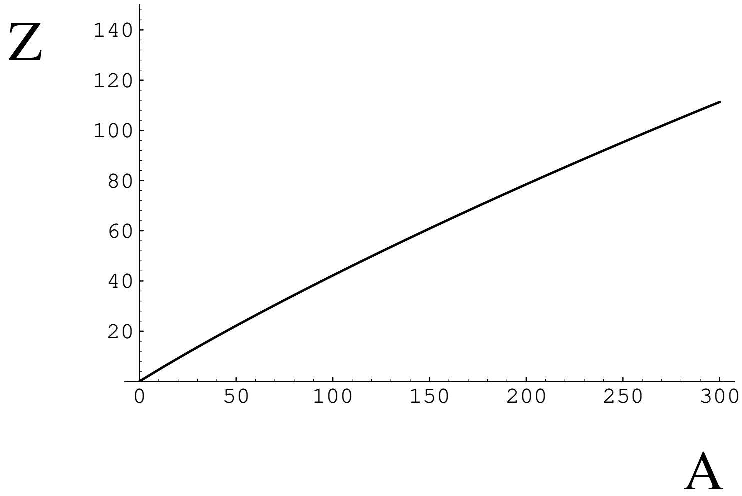

Figure 2: *The value of Z for a given A that maximizes the binding energy per nucleon*

One sees that the the nucleons in fact "live" most of their time outside the It is interesting to examine the whole B/A surface for arbitrary A and Z, see the figure Further, we can look at a contour plot of the B/A, if the 3D plot doesn't do it for you!

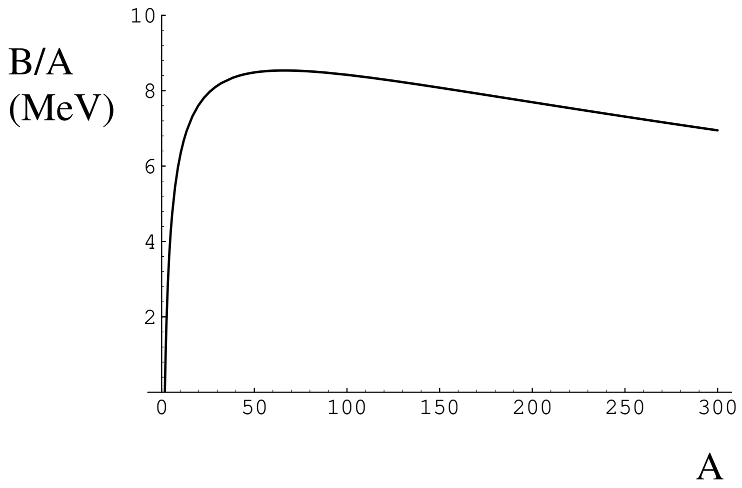

Figure 3: *Binding Energy per Nucleon verses A.*

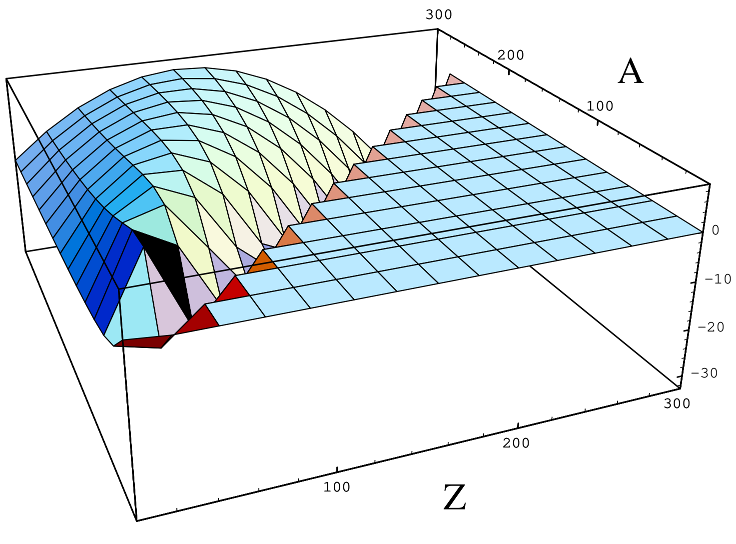

Figure 4: *The Binding energy per nucleon surface against Z and A*

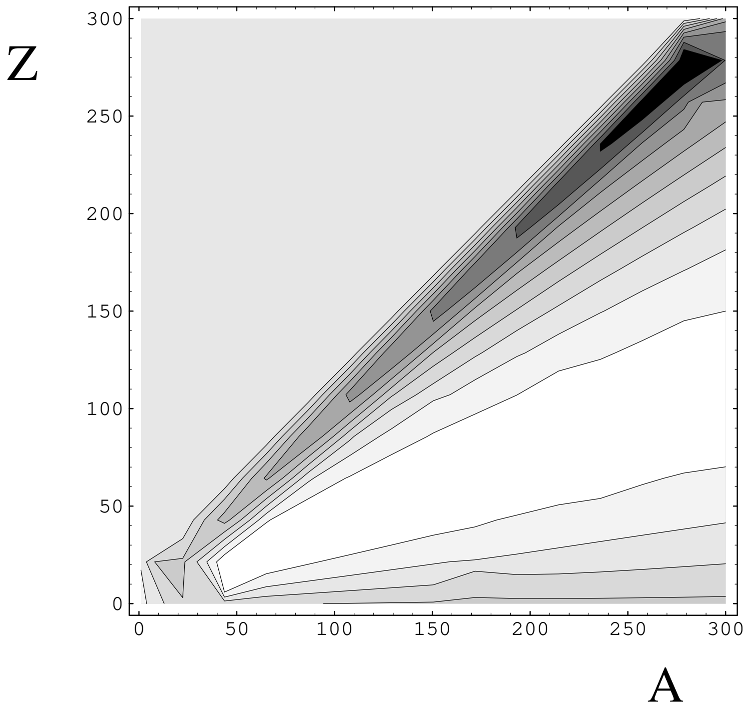

Figure 5: *The Binding energy per nucleon as a contour plot against Z and A.*