PHYS 560: Lectures During Week 4

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October 19, 1999

Abstract

The fourth week of lectures for **Nuclear Physics**, PHY560, Autumn 1999.

October 1999.

1 A Careful Look at the Deuteron

The previous section has taught us that in general the orbital angular momentum of strongly interacting states is not a conserved quantum number due to the presence of non-central forces. These forces commute with the total spin of the system and hence we can classify the states by the J and S quantum numbers. For example, the deuteron is a $S = J = 1$ object, with orbital angular momentum being a mixture of $L = 0$ and $L = 2$. It is therefore convenient to define states of good S, J, L and M_J which are linear combinations of the spatial and spin wavefunctions with different M_S, M_L. The wavefunction for a given system is then a sum over these states of different L. For the deuteron, we need $S = J = 1$ and $L = 0, 2$ for each M_J. The $\mathcal{Y}^M_{JSL}$ by

$$\mathcal{Y}^{M_J}_{JSL} = \sum_{M_S,M_L} \langle S, M_S, L, M_L|J, M_J\rangle\ \chi^{M_S}_S\ Y^{M_L}_L \quad , \tag{1}$$

where $\chi^{M_S}_S$ is the spin wavefunction for the state. For the deuteron in the $M_J = 1$ states the relevant configurations (orthonormal) are

$$\begin{aligned}
&\mathcal{Y}^1_{110} = \langle 11, 00|11\rangle\ \chi^1_1 Y^0_0(\Omega) \\
&= \chi^1_1 Y^0_0(\Omega) \\
&\mathcal{Y}^1_{112} = \langle 1120|11\rangle\ \chi^1_1 Y^0_2(\Omega) + \langle 1021|11\rangle\ \chi^0_1 Y^1_2(\Omega) + \langle 1-122|11\rangle\ \chi^{-1}_1 Y^2_2(\Omega) \\
&= \sqrt{\frac{1}{10}}\ \chi^1_1 Y^0_2(\Omega) - \sqrt{\frac{3}{10}}\ \chi^0_1 Y^1_2(\Omega) + \sqrt{\frac{3}{5}}\ \chi^{-1}_1 Y^2_2(\Omega) \quad .
\end{aligned} \tag{2}$$

A formula that you have seen before, but will be very useful to us is the triple Y^M_L integration formula

$$\int d\Omega\ Y^{M_1}_{L_1}\ Y^{M_2}_{L_2}\ Y^{M_3 *}_{L_3} = \sqrt{\frac{(2L_1+1)(2L_2+1)}{4\pi(2L_3+1)}}\langle L_1 M_1, L_2 M_2|L_3 M_3\rangle\langle L_1 0, L_2 0|L_3 0\rangle \tag{3}$$

We can now compute some angular matrix elements relevant to the deuteron. In order to make progress we rewrite the S_{12} operator in terms of the total spin operator, $S = s_1 + s_2$,

$$\begin{aligned}
&S_{12} \ = \ 3\,(\sigma_1\cdot\hat{\mathbf{r}})\,(\sigma_2\cdot\hat{\mathbf{r}}) \ - \ \sigma_1\cdot\sigma_2 \ = \ 4\,[3\,(s_1\cdot\hat{\mathbf{r}})\,(s_2\cdot\hat{\mathbf{r}}) \ - \ s_1\cdot s_2] \\
&= 2\left[3\,(S\cdot\hat{\mathbf{r}})^2 - S^2\right] \\
&= 2\left[3\left(\frac{1}{2}S_+(\hat{r}_x - i\hat{r}_y) \ + \ \frac{1}{2}S_-(\hat{r}_x + i\hat{r}_y) \ + \ S_z\hat{r}_z\right)^2 - S^2\right] \quad ,
\end{aligned} \tag{4}$$

where we have used the fact that S_{12} is symmetric under interchange $1 \leftrightarrow 2$, and $S_{\pm}$ are the spin raisong and lowering operators. Acting on $|\mathcal{Y}^1_{110}\rangle$ it is easy to show that

$$
\begin{aligned}
\langle \mathcal{Y}^1_{112}|S_{12}|\mathcal{Y}^1_{110}\rangle \;&=\; 2\left(\frac{1}{\sqrt{10}}\sqrt{\frac{4\pi}{5}}\int d\Omega\ Y_2^{0*}Y_2^0Y_0^0 \;+\; 3\sqrt{\frac{3}{10}}\sqrt{\frac{4\pi}{15}}\int d\Omega\ Y_2^{1*}Y_2^1Y_0^0 \right. \\
&\left. + \; 6\sqrt{\frac{3}{5}}\sqrt{\frac{2\pi}{15}}\int d\Omega\ Y_2^{2*}Y_2^2Y_0^0\right) \\
&= \sqrt{8} \quad .
\end{aligned} \tag{5}
$$

We have used

$$
S_{\pm}|1m\rangle = \sqrt{2}|1, m\pm 1\rangle \quad , \tag{6}
$$

and the standard representation of the Y_l^m's.

Similar calculations lead to

$$
\begin{aligned}
\langle \mathcal{Y}^1_{110}|S_{12}|\mathcal{Y}^1_{110}\rangle &= 0 \\
\langle \mathcal{Y}^1_{112}|S_{12}|\mathcal{Y}^1_{110}\rangle &= \sqrt{8} \\
\langle \mathcal{Y}^1_{112}|S_{12}|\mathcal{Y}^1_{112}\rangle &= -2 \quad .
\end{aligned} \tag{7}
$$

The deuteron wavefunction Ψ^M (where M is the magnetic substate) is a linear combination of $L = 0$ and $L = 2$ components

$$
\Psi^M = \cos\omega\ \frac{1}{r}U_0(r)\ \mathcal{Y}^M_{110} + \sin\omega\ \frac{1}{r}U_2(r)\ \mathcal{Y}^M_{112} \quad , \tag{8}
$$

where this relation defines the mixing angle ω between the orthonormal states. The wavefunction satisfies the schrodinger equation for the nucleons moving in the nuclear potential $V(r) = V_{\rm cen} + V_{\rm ten}S_{12}$

$$
\left[-\frac{1}{2\mu}\nabla^2 + V(r)\right]\Psi = -\epsilon\Psi \quad , \tag{9}
$$

where μ is the reduced mass of the N-N system and writing this form out explicitly

$$
\begin{aligned}
&\left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + V_{\rm cen} + V_{\rm ten}S_{12}\right]\cos\omega\ U_0\mathcal{Y}^M_{110} + \left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + V_{\rm cen} + V_{\rm ten}S_{12} + \frac{6}{r^2}\right]\sin\omega\ U_2\mathcal{Y}^M_{112} \\
&= -\epsilon\left[\cos\omega\ U_0\mathcal{Y}^M_{110} + \sin\omega\ U_2\mathcal{Y}^M_{112}\right] \quad .
\end{aligned} \tag{10}
$$

This represents a set of couple equations, which we can obtain by projecting with $\int\ d\Omega\ \mathcal{Y}^M_{112}$ and $\int\ d\Omega\ \mathcal{Y}^M_{112}$ to obtain

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+V_{\rm cen}+V_{\rm ten}\langle 110|S_{12}|110\rangle+\epsilon\right]U_0=-V_{\rm ten}\langle 110|S_{12}|112\rangle\tan\omega\ U_2$$
$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+V_{\rm cen}+\frac{6}{2\mu r^2}+V_{\rm ten}\langle 112|S_{12}|112\rangle+\epsilon\right]U_2=-V_{\rm ten}\langle 112|S_{12}|110\rangle\frac{1}{\tan\omega}\ U_0 \quad (11)$$

These equations have exactly the form we described above. If there are non-zero matrix elements between states of different L, as induced by the tensor force, then the S and D states mix with each other as expected.

We recall that from our naive guestimates made previously, the deuteron is a very extended object compared to the range of the nuclear interaction. In the region outside the potential, the coupled equations become

$$\left[-\frac{1}{2\mu}\frac{d}{dr^2}+\epsilon\right]U_0=0$$
$$\left[-\frac{1}{2\mu}\frac{d}{dr^2}+\frac{6}{r^2}+\epsilon\right]U_2=0 \quad , \qquad (12)$$

which have solutions of the form

$$U_{0,2}\sim\ \left(a+\frac{b}{r}+\frac{c}{r^2}\right)e^{-\gamma r} \quad , \qquad (13)$$

and by direct substitution we find that

$$U_0(r)\rightarrow N_0e^{-\gamma r}$$
$$U_2(r)\rightarrow N_2\left(1+\frac{3}{\gamma r}+\frac{3}{\gamma^2r^2}\right)e^{-\gamma r}$$
$$\gamma=\sqrt{2\mu\epsilon} \quad , \qquad (14)$$

where the $N_{0,2}$ are normalization constants determined solving the equation directly including the potentials.

It is clear that solving the coupled equations is not going to be so easy. In particular, we see that there is the mixing angle ω that is apriori undetermined. We see that for $\omega\ =\ 0$, the Swave wavefunction is uncoupled from the D-wave admixture, but the D-wave is infinitely coupled to the S-wave component. We see that in order to solve these equations, we must compare with data, so that the solution reproduce the binding energy, magnetic moment and quadrupole moment of the deuteron.

In order for us to proceed, we therefore need to know how to relate these wavefunctions to the magnetic and quadrupole moments that are measured.

2 Electromagnetic Interactions.

We have been discussing the admixture of $L = 2$ configurations in the deuteron induced by the tensor component of the one-pion exchange (OPE) potential. Such a component is necessary to explain the electric quadrupole moment possessed by the deuteron. We wish to be more quantitative regarding these and the magnetic moments of multi-nucleon systems and hence we will spend some time setting up some nice relations that hold for given configurations.

Lets start by recalling the multipole expansion of electromagnetic interactions. We consider a charge distribution $\rho(x)$ that has support in a given region of space interacting with an electromagnetic potential that is slowly varying over the volume of the charge distribution. The interaction hamiltonian is given by

$$\begin{aligned} H_{\mathrm{int}} &= \int d^3x \; j_\mu(x) A^\mu(x) \\ &= \int d^3x \; [\rho(x)\phi(x) - \mathbf{j}(x) \cdot \mathrm{A}(x)] \\ &= \int d^3x \; \Big[\rho(x)\left(\phi(0) \;+\; \partial_i\phi(0)x^i \;+\; \frac{1}{2}\partial_i\partial_j\phi(0)x^i x^j \;+\; ...\right) \\ &\quad -\mathbf{j}(x)\left(\mathbf{A}(0) \;+\; \partial_i\mathbf{A}(0)x^i \;+\; \frac{1}{2}\partial_i\partial_j\mathbf{A}(0)x^i x^j \;+\; ...\right)\Big] \quad . \end{aligned} \tag{15}$$

Recalling the $\mathbf{E}^j = -\partial^j\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ we have that

$$H_{\mathrm{int}} = Q\phi(0) - \mathbf{P}\cdot\mathbf{E} - \frac{1}{6}Q^{ij}\partial^i\mathbf{E}^j - \mathbf{M}\cdot\mathbf{B} \;+\; ... \quad , \tag{16}$$

where the moments are defined by

$$\begin{aligned} \mathbf{P} &= \int d^3x \; \rho(x)\mathbf{x} \\ \mathbf{M} &= \frac{1}{2}\int d^3x \; \mathbf{x} \times \mathbf{j}(x) \end{aligned}$$

$$Q^{ij} = \int d^3x \left(3x^i x^j - \delta^{ij} x^2\right) \rho(x) \quad , \tag{17}$$

and so forth.

2.1 Magnetic Moments

We are interested in asking what these static moments are for systems of nucleons. In order to answer this question we need to break down the contribution into parts arising from the orbital motion of charges (giving a contribution from the velocity of the proton charge) and a part coming from the intrinsic spin of the nucleons. Let us begin by forming the contribution to the magnetic moment arising from the orbital motion of the proton charge. The current density of a charge distribution with velocity $\mathbf{v}$ is given by $\mathbf{j}(x) = \rho(x)\mathbf{v}(x)$. Therefore, our definition of $\mathbf{M}$ leads to

$$\begin{aligned}
\mathbf{M}_{\rm orb} &= \frac{1}{2} \int d^3x \; \mathbf{x} \times \mathbf{j} \\
&= \frac{1}{2} \int d^3x \; \rho(x) \mathbf{x} \times \mathbf{v} \\
&= \frac{e}{2M_N} \int d^3x \; \rho'(x) \mathbf{x} \times \mathbf{p} \\
&= \frac{e}{2M_N} \int d^3x \sum_{j=1}^{Z} \psi^* \, l^{(j)} \psi \quad ,
\end{aligned} \tag{18}$$

where ρ' is the probability density associated with the charge density ρ, ψ is the wavefunction of the state and $l^{(j)}$ is the orbital angular momentum operator. Further, being good quantum mechanics we realize that forming an operator will be the way to go and we can define

$$\hat{\mathbf{M}}_{\rm orb} = \mu_0 \sum_{j=1}^{Z} l^{(j)} \quad , \tag{19}$$

where $\mu_0 = e/(2M_N)$ is the Dirac moment of a nucleon. It is convenient to decompose this object into it irreducible representations of isospin as we will be using isospin to classify nuclear states, and recalling that a proton was projected from a nucleon spinor by $p = \frac{1}{2}(1 + \tau_3)N$ and similarly and neutron $n = \frac{1}{2}(1 - \tau_3)N$. We can then write

$$\hat{\mathbf{M}}_{\rm orb} = \mu_0 \sum_{j=1}^{A} \frac{1}{2}(1 + \tau_3^{(j)}) l^{(j)} \quad , \tag{20}$$

where the sum has been extended from only a sum over protons to a sum over all nucleons in the system. The electric charge of each is projected out with the isospin operator.

It is easy to include the contribution to the magnetic moment from the intrinsic magnetic moments of the neutron and proton. You will recall the the magnetic moment of the proton is $\mu_p = +2.792847386 \pm 0.000000063\ \mu_0 = \kappa_p \mu_0$ and magnetic moment of the neutron is $\mu_n = -1.91304275 \pm 00000045 \mu_0 = \kappa_n \mu_0$. Recalling that there is a "g-factor" of 2 for spin $\frac{1}{2}$ particles we have that

$$
\begin{aligned}
\hat{\mathbf{M}}_{\rm spin} &= 2\mu_0 \left(\sum_{j=1}^{Z} \kappa_p S^{(j)} + \sum_{j=Z+1}^{A} \kappa_n S^{(j)} \right) \\
&= 2\mu_0 \sum_{j=1}^{A} \left(\frac{1}{2}(1+\tau_3^{(j)}) \left[\frac{1}{2}(\kappa_p + \kappa_n) + \frac{1}{2}(\kappa_p - \kappa_n) \right] S^{(j)} \right. \\
&\left. + \frac{1}{2}(1-\tau_3^{(j)}) \left[\frac{1}{2}(\kappa_n + \kappa_p) + \frac{1}{2}(\kappa_n - \kappa_p) \right] S^{(j)} \right) \\
&= \mu_0 \left((\kappa_p + \kappa_n) S + (\kappa_p - \kappa_n) \sum_{j=1}^{A} \tau_3^{(j)} S^{(j)} \right) \quad , \qquad (21)
\end{aligned}
$$

where S is the total spin operator. We sum the orbital with with spin contribution to find the total magnetic moment operator $\hat{\mathbf{M}} = \hat{\mathbf{M}}_{\rm spin} + \hat{\mathbf{M}}_{\rm orb}$.

Let me make some comments here. We have made some big assumptions getting to this point. Firstly, we have assumed that the only charged currents in the nucleus are those from the motion of the nucleons. We have neglected the contributions from mesons, which we know are present in a field theory description of nucleons, and as we saw last time are responsible for the long-range part of the interaction. Secondly, we have assumed that the nucleons retain their free-space "identity", by which I mean that their magnetic moments in a nuclear environment are the same as those in free-space. Corrections to this are a bit harder as we can only ever measure S-matrix elements and not the individual contributions. Clearly, these assumptions will be exact in the limit that the nuclear interactions "turn-off", as they then would not "know" about each other. So one might guess that corrections to these results behave like $E_{\rm binding}/M_N$. I will leave this subject at this point, to return to later. This is just a warning, NOT to stop thinking about this!

We will be wanting to form matrix elements of the above operators between states of good J and S, and not good L (tensor type interactions). In order to proceed we need to know how to take such matrix elements and we will need a simple application of the Wigner-Eckhart theorem, that we will briefly derive.

The Wigner-Eckhart theorem tells us that the matrix element of a tensor operator $T_k^{(\mu)}$ of rank $2k+1$ has matrix elements between states of good total angular momentum J of the form

$$\langle J'm'|T_K^{(\mu)}|Jm\rangle = \langle JmK\mu|J'm'\rangle\langle J'||T_K||J\rangle \quad . \tag{22}$$

This demonstrates that the matrix element can be decomposed into a reduced matrix element (denoted with the double vertical lines) , that is independent of the alignment of the states in the subspace, and a Clebsch-Gordan coefficient that factors in the orientation in the subspace. If we have an arbitrary vector operator, i.e. $k = 1$, then this becomes

$$\langle J'm'|T_1^{(\mu)}|Jm\rangle = \langle Jm1\mu|J'm'\rangle\langle J'||T_1||J\rangle \quad , \tag{23}$$

and hence in the ratio of matrix elements for two different vector operators with same alignment between states of the same alignment, the Clebsch-Gordan factor cancels

$$\frac{\langle J'm'|A^{(\mu)}|Jm\rangle}{\langle J'm'|B^{(\mu)}|Jm\rangle} = \frac{\langle J'||A_1||J\rangle}{\langle J'||B_1||J\rangle} \quad , \tag{24}$$

where I have dropped the subscript denoting vector operator. Imagine now that one of the operators is the angular momentum generator itself

$$\begin{aligned}
&\langle J, m = J|J^z|J, m = J\rangle = \langle JJ10|JJ\rangle\langle J||J||J\rangle \\
&J = \sqrt{\frac{J}{J+1}}\langle J||J||J\rangle \\
&\langle J'||J||J\rangle = \sqrt{J(J+1)}\ \delta_{JJ'} \quad .
\end{aligned} \tag{25}$$

To continue, we note that the matrix element of a scalar operator formed by contracting the angular momentum generators with a vector operator A, must be proportional to the reduced matrix element of A. We can then determine the constant of proportionality by setting $A = J$ as follows

$$\begin{aligned}
&\langle Jm' = J|J\cdot A|Jm = J\rangle = C\ \langle J||A||J\rangle \\
&J(J+1) = C\ \langle J||J||J\rangle\ =\ C\sqrt{J(J+1)} \\
&\langle Jm' = J|J\cdot A|Jm = J\rangle = \sqrt{J(J+1)}\langle J||A||J\rangle \ .
\end{aligned} \tag{26}$$

Hence, it follows from the above that

$$\langle Jm' = J|A_z|Jm = J\rangle = \frac{\langle Jm' = J|J\cdot A|Jm = J\rangle}{J(J+1)}\langle Jm' = J|J_z|Jm = J\rangle$$
$$= \frac{J}{J(J+1)}\langle Jm' = J|J\cdot A|Jm = J\rangle \quad . \tag{27}$$

As an application of this result to the magnetic moment, consider an operator of the form

$$\mathcal{O} = a\sum_{j=1}^{A} l^{(j)} + b\sum_{j=1}^{A} s^{(j)} + c\sum_{j=1}^{A}\tau_3^{(j)} l^{(j)} + d\sum_{j=1}^{A}\tau_3^{(j)} s^{(j)} \quad , \tag{28}$$

where l and s are the orbital and spin angular momentum generators, and the constants a,b,c,d are known. Matrix elements of operators involving τ_3 will depend upon the I_z quantum number of the state acted on. Therefore if we add the matrix elements between states of equal and opposite I_z (denoted by $\langle M_\pm\rangle$), theses contributions vanish, leaving only contributions from a and b,

$$\langle M_+\rangle + \langle M_-\rangle = \langle J, m = J, I, I_Z|\mathcal{O}|J, m = J, I, I_Z\rangle + \langle J, m = J, I, -I_Z|\mathcal{O}|J, m = J, I, -I_Z\rangle$$
$$= 2\langle J, m = J, I|a\sum_{j=1}^{A} l^{(j)} + b\sum_{j=1}^{A} s^{(j)}|J, m = J, I\rangle$$
$$= 2\langle J, m = J, I|\frac{1}{2}(a+b)\sum_{j=1}^{A}(l^{(j)} + s^{(j)}) + \frac{1}{2}(a-b)\sum_{j=1}^{A}(l^{(j)} - s^{(j)})|J, m = J, I\rangle \quad . \tag{29}$$

The first operator is the total angular momentum of the state $\sum_{j=1}^{A}(l^{(j)} + s^{(j)}) = J$, while the second requires use of the relation we proved above. Looking at the $z-$component

$$\langle J, m = J, I|\sum_{j=1}^{A}(l^{(j)} - s^{(j)})_z|J, m = J, I\rangle = \langle J, m = J, I|(L-S)_z|J, m = J, I\rangle$$
$$= \frac{1}{J+1}\langle J, m = J, I|J\cdot(L-S)|J, m = J, I\rangle \quad , \tag{30}$$

and so we have that

$$\langle M_+\rangle + \langle M_-\rangle = (a+b)J + \frac{1}{J+1}\langle J, m = J, I|J\cdot(L-S)|J, m = J, I\rangle \tag{31}$$

and by using $S^2 = (J-L)^2$ and $L^2 = (J-S)^2$ we find that

$$\langle M_+\rangle + \langle M_-\rangle = (a+b)J + \frac{a-b}{J+1}\langle J, m=J, I|L^2 - S^2|J, m=J, I\rangle \quad (32)$$

Applying this relation to the magnetic moments of Mirror nuclei, e.g. ^{25}Mg and ^{25}Al, where the nuclei only differ by the nucleon on top of the ^{24}Mg closed shell being either a proton or a neutron (the nuclear wavefunctions are identical except for the I_z quantum number) ,

$$\mu_+ + \mu_- = \mu_0\left[\left(\frac{1}{2}+\kappa_p+\kappa_n\right)J + \left(\frac{1}{2}-\kappa_p-\kappa_n\right)\frac{\langle L^2-S^2\rangle}{J+1}\right] \quad . (33)$$

In forming this relation we have used

$$a = \frac{1}{2}\mu_0 \quad , \quad b = (\kappa_p+\kappa_n)\,\mu_0 \quad . \quad (34)$$

Lets get back to our nucleus of interest for the moment, the deuteron. We want to find out what we expect its magnetic moment is for the possible spin-space wavefunctions that it can possible have. It is clear from eq. (33) that is magnetic moment is ($J=1$, $I=0$)

$$\mu_d = \frac{\mu_0}{2}\left[\left(\frac{1}{2}+\kappa_p+\kappa_n\right) + \left(\frac{1}{2}-\kappa_p-\kappa_n\right)\frac{\langle L^2-S^2\rangle}{2}\right] \quad . \quad (35)$$

If the deuteron had space-spin configurations given by $^{2s+1}L_J$, then we find that

$$\begin{aligned}
{}^1P_1 &: \mu_d = \frac{1}{2}\,\mu_0 \\
{}^3S_1 &: \mu_d = \kappa_p + \kappa_n = 0.879\ \mu_0 \\
{}^3P_1 &: \mu_d = \frac{1}{2}\left(\frac{1}{2}+\kappa_p+\kappa_n\right)\ \mu_0 = 0.689\ \mu_0 \\
{}^3D_1 &: \mu_d = \frac{1}{2}\left(\frac{3}{2}-\kappa_p-\kappa_n\right)\ \mu_0 = 0.310\ \mu_0 \quad .
\end{aligned} \quad (36)$$

As we know that the deuteron wavefunction is a linear combination of S and D states, we realize that the contributions to the magnetic moments will add incoherently, as they are in a different partial wave and this, for a mixing angle ω as defined last time we have that

$$\mu_d = \left(0.879\cos^2\omega \ + \ 0.310\sin^2\omega\right)\ \mu_0 \ = \ \left(0.879 - 0.569\sin^2\omega\right)\ \mu_0 \quad (37)$$

and in order to reproduce the experimentally observed $\mu_d = 0.857$ we require that $\sin^2\omega \sim 4\%$. If the deuteron was a combination of the P-states, then there would be an interference term (dependent upon the exact form of the radial wavefunction) as they are the same partial wave.

Notice that the result is independent of the form of the radial wavefunction. This is the most naive estimate of the D-state admixture, in the most naive single particle model of the nucleus. There are corrections arising from meson exchange, for example, we might imagine attaching a photon to any lines in the graph we drew last time responsible for the potential between nucleons. The graphs we are keeping here are those where the photon attaches to a nucleon, but there are other graphs we have not retained where the photon attaches to the exchanged meson.

2.2 Quadrupole Interactions.

You will recall from last time that we defined the electric quadrupole interaction be performing a taylor expansion on the electromagnetic four-potential and identifying the term that transformed as a two index symmetric traceless tensor under rotations,

$$\begin{aligned} H_{\rm int}^{L=2} &= -\frac{1}{6}\ Q^{ij}\ \frac{\partial E^i}{\partial x_j} \\ Q^{ij} &= \int\ d^3x\ \rho(x)\ (3x^ix^j - x^2\delta^{ij}) \quad , \end{aligned} \tag{38}$$

where $\rho(x)$ is the electric charge density of the localized object, in this case the nucleus. It is clear that this is a $L = 2$ object by inserting $i = j = z$, in which case we find $Q^{zz} \sim Y_2^0(\Omega)$.

Just as we did for the magnetic moment interaction and hence defining the magnetic moment of a nucleus wish to use eq. (38) to define the quadrupole moment of a nucleus. Let us choose to evaluate the diagonal matrix element of Q^{zz} between states defined by $|J, M = J\rangle$ (summing over all the electrically charged objects inside the nucleus)

$$\begin{aligned} Q = \langle J, J|Q^{zz}|J, J\rangle &= \sum_{k=1}^{Z}\ \int\ d^3x\ \rho^{(k)}(x)\ (3z^{(k)}z^{(k)} - (r^{(k)})^2) \\ &= \sum_{k=1}^{Z}\ \int\ d^3x\ \rho^{(k)}(x)\ (r^{(k)})^2\sqrt{\frac{16\pi}{5}}\ Y_2^0(\Omega^{(k)}) \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{16\pi}{5}} \int d^3x \; \Psi^*_{J,J} \sum_{k=1}^{Z} e \; (r^{(k)})^2 \; Y_2^0(\Omega^{(k)}) \; \Psi_{J,J} \\
&= \sqrt{\frac{16\pi}{5}} \langle Q_2 \rangle \quad .
\end{aligned} \tag{39}
$$

This sets the conventional definitions of Q and Q_2. This also leads to the generalization to arbitrary multipoles Q_λ, where λ denotes the multipolarity,

$$
Q_\lambda = \sum_{k=1}^{Z} e \; (r^{(k)})^\lambda \; Y_\lambda^0(\Omega^{(k)}) \quad . \tag{40}
$$

We still have to define the operator Q^{ij}, which we do in terms of Q. We note that we can use similar arguments that lead us to the form for the magnetic moment operator. We realize that acting in the space of states defined by $|J,M\rangle$, the matrix element of any operator in this space must be reproduced by an analogous operator formed from the J_i. The quadrupole operator is a two index, symmetric, traceless tensor, and hence the operator given in terms of the J_i must also have these properties. Therefore we find the general form for Q^{ij} must be

$$
\langle J,M|Q^{ij}|J,M\rangle = C \; \langle J,M|\frac{3}{2}\left(J^iJ^j + J^jJ^i\right) - \delta^{ij}J^2|J,M\rangle \quad , \tag{41}
$$

where C is a yet to be determined constant. It is again convenient to examine the matrix element of Q^{zz} between states of $|J, M = J\rangle$

$$
\begin{aligned}
&\langle J,J|Q^{zz}|J,J\rangle = C \; \langle J,J|\frac{3}{2}\left(J^zJ^z + J^zJ^z\right) - J^2|J,J\rangle \\
&= C \; 3J^2 - J(J+1) \\
&= C \; J \; (2J-1) = Q \quad ,
\end{aligned} \tag{42}
$$

and hence we have solved for the constant

$$
C = \frac{Q}{J(2J-1)} \quad . \tag{43}
$$

We can then write the operator as

$$
Q^{ij} = \frac{Q}{J(2J-1)} \left[\frac{3}{2}\left(J^iJ^j + J^jJ^i\right) - \delta^{ij}J^2\right] \quad . \tag{44}
$$

A couple of general comments are worth making at this point. Notice that the charge density is the square modulus of the wavefunction and as such it is not possible for them to have odd-electric moments if our world respects time reversal invariance. This is most easily seen by considering the time reversal properties of an odd number of J_i combined together into a traceless, symmetric tensor. This clearly has the same property under time-reversal as a single J_i, which is odd. Combined with the fact that E_j is even under time-reversal, we find that such odd-electric moments are forbidden by Time-Reversal Invariance which we know is pretty well respected in nature (i.e. we will forget about it!).

Also, we note that a nucleus of spin J can have a static moment of highest multipolarity $2J$ (i.e. $J \otimes J = \lambda$). This means that a $J = 0, \frac{1}{2}$ object cannot have an electric quadrupole moment but a $J = 1$ object such as the deuteron can.

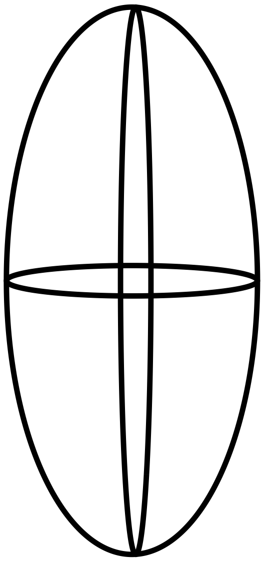

Figure 1: *A non-spherical object with a quadrupole moment.*

It is worth spending a few minutes to get a physical grip on what a quadrupole moment is (I hope I am not boring you). Imagine we have a ellipsoidal object with semi-major axis of length "a" and a semi-minor axis of length "b". Lets us start by computing the charge in terms of the charge density ρ which we will assume to be uniform 9 out to the nuclear surface and then falls abruptly to zero), and the lengths of axes a, b. You will recall that the trick for performing volume integrals over ellipsoids is to rescale the axis by the length of the semi-major or -minor axis, e.g $z' = z/a, x' = x/b$ and $y'y/b$. We then have that

$$\begin{aligned} Z &= \int d^3x \ \rho(x) \\ &= ab^2 \int d^3x' \ \rho(x') \\ &= ab^2\rho\frac{4\pi}{3} \quad . \end{aligned} \tag{45}$$

Next, lets consider the quadrupole moment of this object where we define the z-axis to be the symmetry axis along the semi-major direction, i.e. the axis of cylindrical symmetry. This is called the body-axis, and I will denote coordinates in this frame with a subscript b, i.e. z_b. It is straightforward to compute the quadrupole moment $Q^{z_b z_b}$ in this frame,

$$\begin{aligned} Q^{z_b z_b} &= \int d^3x_b \ \rho(x_b) \ (3z_b^2 - r_b^2) \\ &= \int d^3x_b \ \rho(x_b) \ (2z_b^2 - x_b^2 - y_b^2) \\ &= eab^2 \int d^3x_b \ \rho(x_b) \ (2a^2 z_b'^2 - b^2 x_b'^2 - b^2 y_b'^2) \\ &= 2eab^2(a^2 - b^2) \int d^3x_b \ \rho(x_b) \ z_b'^2 \\ &= \frac{2}{5} Ze(a^2 - b^2) \quad , \end{aligned} \tag{46}$$

where we have used the fact that in the x' coordinates the ellipsoid is a sphere, and set $\langle x'^2 \rangle = \langle y'^2 \rangle = \langle z'^2 \rangle$. Further we can show that the quadrupole moment about any axis, we call z, corresponding to a rotation from the body-axis z_b by an angle θ is given by

$$\begin{aligned} Q^{zz} &= \int d^3x \ \rho(x) \ (3z^2 - r^2) \\ &= \int d^3x_b \ \rho(x_b) \ (2(z_b \cos\theta + x_b \sin\theta)^2 - (x_b\cos\theta - z_b \sin\theta)^2 - y_b^2) \\ &= \frac{1}{2}\left(3\cos^2\theta - 1\right) Q^{z_b z_b} \quad , \end{aligned} \tag{47}$$

where we have again used symmetry properties of the integration.

Our discussion of this object has been entirely classical up to this point. However, we are interesetd in asking about quantum systems, and states defined by J and M_J. The quantum nature of the nuclei tells us that its angular momentum J and its angular orientation are conjugate variables, and if we knew that the ellipsoid was aligned with its semi-major axis along the quantization axis, then we would not know what its angular momentum is! What this means is that there is a minimum angle between the body-axis and the quantization axis which we can determine by a hand waving argument! The maximum J_Z is equal to J, while the magnitude of the angular momentum vector is $\sqrt{J(J+1)}$, leading to a minimum angle between the axes of $\cos\theta_{\rm min} = \sqrt{\frac{J}{J+1}}$. Therefore, by our previous definitions, the observable quadrupole moment of this ellipsoidal object, is related to its quadrupole moment as computed along its body-axis by

$$Q = \frac{2J-1}{2(J+1)} Q_b \quad . \tag{48}$$

Classically, we can have a charge distribution with an quadrupole deformation, that has vanishing rotation, so one would have guessed that in fact a $J = 0, \frac{1}{2}$ object would have had a quadrupole moment. In fact, it is a quantum mechanical effect that such an object does not. It is the uncertainty principle and the fact that the angular momentum and angle are conjugate variables. One cannot make a rank two tensor from 0 or 2 magnetic substates. We see that in fact a $J = 0$ and a $J = \frac{1}{2}$ nucleus could have a large quadrupole deformation, but we would not be able to observe it in any experiment. However, we might imagine spinning up the nucleus and observing a characteristic $J(J+1)$ type rotator spectrum, and by measuring transition strengths of γ decays, and also the moment of inertia (from the energy level spacings) we could determine the quadrupole moment that is independent of J. This is done in light nuclei and one can conclude that the $J = \frac{1}{2}$ level does in fact have a quadrupole moment, but it can never be observed. The question to ask is if a tree falls and no one is around does it make a noise, and my answer to this is who cares! If I don't here it I really don't care. Same thing with this quadrupole moment. It is interesting in principle, but who cares if it can't be measured and has no consequences.

3 Quadrupole Moment of the Deuteron

The quadrupole moment of the deuteron is $Q_d = 0.2860 \pm 0.0015 \mathrm{fm}^2$. Writing the wavefunction of the deuteron as we have before as

$$\Psi_d^m = \cos\omega \ \frac{1}{r} U_0(r) \mathcal{Y}_{101}^m + \sin\omega \ \frac{1}{r} U_2(r) \mathcal{Y}_{121}^m \quad , \tag{49}$$

we have that

$$\begin{aligned} Q &= \sqrt{\frac{16\pi}{5}} \int d^3x \ \left(\frac{r}{2}\right)^2 \ Y_2^0(\Omega) \\ &= \sqrt{\frac{\pi}{5}} \int dr \ r^2 \ d\Omega \ r^2 Y_2^0(\Omega) \ \frac{1}{r^2} \ \left[\sin^2\omega \ \mathcal{Y}_{121}^{1*} \mathcal{Y}_{121}^1 |U_2(r)|^2 + 2 \sin\omega \cos\omega \mathcal{Y}_{121}^{1*} \mathcal{Y}_{101}^1 U_0(r) U_2(r)\right] \end{aligned} \tag{50}$$

You will recall that

$$\begin{aligned} \mathcal{Y}_{121}^1 &= \sqrt{\frac{3}{5}} Y_2^2 \chi_1^{-1} - \sqrt{\frac{3}{10}} Y_2^1 \chi_1^0 + \sqrt{\frac{1}{10}} Y_2^0 \chi_1^1 \\ \mathcal{Y}_{101}^1 &= Y_0^0 \chi_1^1 \quad , \end{aligned} \tag{51}$$

which leads to

$$Q = \frac{1}{5\sqrt{2}} \int dr r^2 \left[\sin\omega \cos\omega \ U_0(r) U_2(r) - \frac{1}{\sqrt{2}} |U_2(r)|^2 \sin^2\omega\right] \tag{52}$$

We notice that a pure D-state has a $-ve$ quadrupole moment. It is clear that we must have a non-trivial amount of S-D mixing in order to obtain a $+ve$ quadrupole moment as is observed for the deuteron. We recall that the deuteron magnetic moment indicated $\sin^2\omega \sim 0.04$. Inserting this value into eq. (52) and using the measured value of Q_d we find that $\int \ dr \ r^2 U_0 U_2 \sim 10 \mathrm{fm}^2$, entirely consistent with the size of the deuteron we have determined from scattering.

We should also note at this stage that it might appear that the quadrupole moment is a better observable from which to extract $\sin\omega$. The magnetic moment contribution from the D state is small and hence yields a large uncertainty compared to Q_d. However, you will also recall that the magnetic moment was independent of the radial wavefunctions and hence the nucleon-nucleon potential. This is not true of the quadrupole moment that depends directly on both the S and D wave radial wavefunctions.

References