PHYS 560: Lectures During Week 3

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Abstract

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1 A Simple Minded Look at the Deuteron

The deuteron is the lightest, nontrivial nucleus (where I am viewing the proton as the lightest nucleus). It is a bound state of a neutron and proton, and one can determine its binding energy to be $E = 2.224644 \pm 0.000034$MeV. In addition to the binding energy one can measure electromagnetic quantities, such as the magnetic moment $\mu_d = 0.857406 \pm 0.000001$N.M. and a quadrupole moment of $Q_d = 0.2860 \pm 0.0015\mathrm{fm}^2$. Let me just make a comment in passing here, we will return for a closer look later. The non-zero quadrupole moment tells us directly that the deuteron is not just and n-p bound into an S-wave. An S-wave object has spherical symmetry and therefore cannot have a quadrupole deformation (component of the shape $\sim Y_{20}$) and hence electric quadrupole moment.

You will recall that the spatial extent of a wavefunction is determined by its binding energy and not by the radius of the potential. The small binding energy would indicate that the deuteron is an extended object and that the details of the potential causing its formation are not really that relevant to its properties. Further, as the deuteron is the only bound state of the NN system, it is going to be an S-state (neglecting the mixing of different L-states) as we know from quantum mechanics that the higher angular momentum states are at higher energy.

Noting that the deuteron has a non-zero magnetic moment which would indicate that it is $S=1$ and not $S=0$, and also that the scattering length in the $S = 1$ channel is $+ve$ and not $-ve$ as shown in the figure, we will use the square well as given in eq. (1) (which reproduce the scattering lengths and effective rangein both channels)

$$
\begin{aligned}
S=1 \quad &: \quad V^{(1)}_{\mathrm{square}} = 38.5 \text{ MeV} \quad , \quad R^{(1)}_{\mathrm{square}} = 1.93\mathrm{fm} \quad ; \quad (VR^3)^{(1)}_{\mathrm{square}} = 1.40 \text{ fm}^2 \\
S=0 \quad &: \quad V^{(0)}_{\mathrm{square}} = 14.3 \text{ MeV} \quad , \quad R^{(0)}_{\mathrm{square}} = 2.50\mathrm{fm} \quad ; \quad (VR^3)^{(0)}_{\mathrm{square}} = 1.13 \text{ fm}^2
\end{aligned} \tag{1}
$$

to look for a bound state.

We look for a wavefunction of the form

$$
\psi_d(\mathbf{r}) = \frac{1}{r} u_0(r) Y_{00}(\Omega) \quad , \tag{2}
$$

that is a solution to the schrodinger equation

$$
\left[-\frac{\nabla^2}{2\mu} + V(r) \right] \psi = -\epsilon \psi \quad , \tag{3}
$$

where ϵ is a $+ve$ number. The $-ve$ sign indicates we are looking for a bound state, with energy less than a zero-energy incident scattering wave. The potential is a square well as given in eq. (1) and μ is the reduced mass of the system $\mu = M_N/2$. It is easy to show that eq. (3) leads to an equation for the radial part of the wavefunction of the form

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + V + \epsilon\right] u_0(r) = 0 \quad . \tag{4}$$

When $V(r)$ is a discontinuous step potential we can solve this analytically without having to go to Mathematica.

In the region inside the range of the potential, which I will call R, with $V = -V_0$ (I hope the notation is not getting out of hand) the wavefunction behaves sinusoidally and

$$u_<(r) = A \ \sin(K_d r) \tag{5}$$

where K_d is the momentum vector inside the well

$$K_d = \sqrt{2\mu(V_0 - \epsilon)} \quad . \tag{6}$$

Outside the range of the potential the wavefunction $u_0(r)$ dies off exponentially (after excluding the unphysical exponentially growing solution)

$$u_>(r) = Be^{-\kappa_d r} \quad , \tag{7}$$

where

$$\kappa_d = \sqrt{2\mu\epsilon} \quad . \tag{8}$$

At the boundary of the potential these solutions in the two regions must join up continuously and smoothly. It is useful to form the logarithmic derivative of the wavefunction so as to remove the normalization constants A and B from consideration, and the continuity requirement can be written as an equality between the logarithmic derivatives at the boundary. This relation provides a constraint on ϵ,

$$\begin{aligned}
&\left(\frac{1}{u_<}\frac{du_<}{dr}\right)_{r=R} = \left(\frac{1}{u_>}\frac{du_>}{dr}\right)_{r=R} \\
&K_d \cot(K_d R) = -\kappa_d \\
&\cot(K_d R) = -\frac{\kappa_d}{K_d} = -\sqrt{\frac{\epsilon}{V_0 - \epsilon}} \sim -\sqrt{\frac{2.22}{36.2}} \quad .
\end{aligned} \tag{9}$$

It is clear that in order to have a small binding energy energy, the wave number must satisfy $K_d R \sim \pi/2$ and hence

$$2\mu V_0 R^2 = \frac{\pi^2}{4} \quad , \tag{10}$$

and that no bound state is possible unless $2\mu V_0 R^2 > \frac{\pi^2}{4}$. Numerically we find that $M_N (V R^2)^{(1)}_{\rm square} = 3.46$ which is larger than $\pi^2/4 = 2.47$

I have plotted the numerical solution for the bound states to the $S = 1$ square well potential given in eq. (**??**) , along with the $E = 0$ scattering solution in Fig. (**??**). One sees that the the nucleons in fact "live" most of

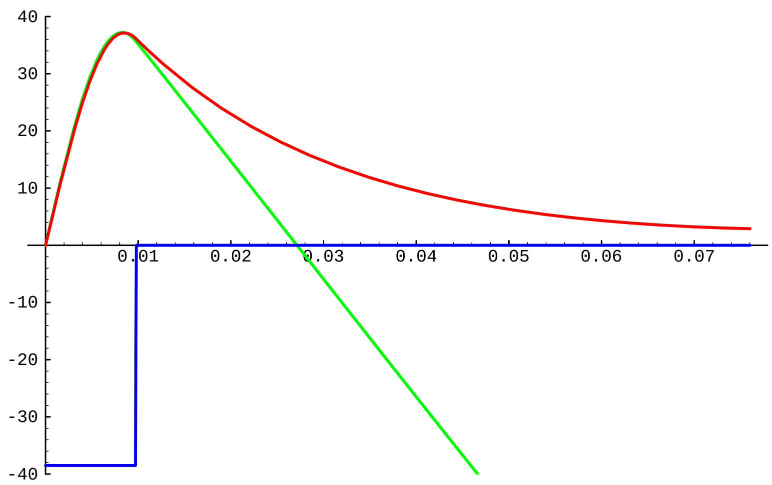

Figure 1: *Square well and bound state wavefunction in* $S = 1$ *channel.*

their time outside the range of the potential binding them together! Further, they live at distances that are large compared to the scattering length in this channel. It is clear how the scattering length and bound state are related, as they both satisfy the continuity condition at the boundary of the potential.

2 One Pion Exchange Potential.

During the last lecture we looked at N-N scattering in the low-energy regime, and determined what square well potentials were required to reproduce the data. We found that the potential had very different depths and ranges, but had a volume integral that was approximately the same. Also, for homework this week you are looking at the low energy phase shifts induced by a Yukawa potential of range $1/M_\pi$ and coupling of $\alpha = 0.069934$. The phase shift will be the same for spin triplet and singlet channels by construction.

In the extreme low energy limit, we would expect that the long-range part of the potential between nucleons is dominated by the exchange of the lightest strongly interacting particle (as we are requiring all interactions are local), which we know to be the pion. Let us consider the dynamics of non-relativistic nucleons with π's. As we are dealing with low energy processes, with a characteristic momentum transfer of order $E \sim q^2/M_N$, we expect that virtual pair production of nucleons is suppressed by $\sim q^2/(2M_N)^2$ and that π production doesn't occur for energies $\sim< M_\pi$. Therefore, we can use, honest to god, two-component spinors to describe the theory and use a lagrange density of the form

$$\begin{aligned}\mathcal{L}_{\mathrm{n.r.}} &= N^\dagger\left(i\partial_t - \frac{\nabla^2}{2M_N}\right)N - g_A N^\dagger\sigma\cdot\Pi\ N \\ &-\frac{1}{2}C_s N^\dagger N N^\dagger N - \frac{1}{2}C_t N^\dagger\sigma^b N N^\dagger\sigma^b N \quad + \quad .... \quad ,\end{aligned} \tag{11}$$

where we have not included the two-π fundamental vertex from chiral symmetry and $g_A = 1.25$ is the axial coupling constant and

$$\Pi = \frac{1}{\sqrt{2}f_\pi}\tau^a\nabla\pi^a \quad + \quad .... \quad , \tag{12}$$

where the π^a are the pion fields e.g. $\pi^3 = \pi^0$. The pion decay constant is $f_\pi = 132\mathrm{MeV}$. It is clear from the kinetic energy part of this lagrangian that the Euler-Lagrange give rise to the Schrodinger equation as the classical trajectory in field space, as required. The four-nucleon operators have unknown coefficients C_s, C_t, that have to be fit to data. We will not worry about these terms at present, if at all in this course. As they are local objects, they give rise to delta function interactions $\sim \delta^4(r-r')$, on the scale of the long-wavelength interactions we are considering

We are interested in the long-range behaviour of the nuclear force and this will be induced by the exchange of π's. Let us construct the potential arising from one-pion exchange (OPE), it arises from the tree-level graphs with four external nucleons and an off-shell pion, as shown in fig. (**??**).

The matrix element for nucleons scattering by pion exchange is given by the graph, using our usual Feynman rules, at lowest order in perturbation

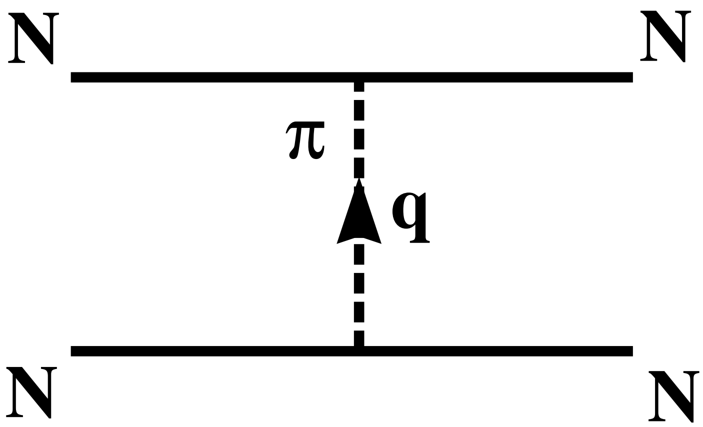

Figure 2: *Feynman diagram that generates the one-pion exchange potential.*

theory (and recalling that a derivative acting on a scalar field with incoming momentum q gives $\partial\phi = -iq\phi$)

$$\begin{aligned}\mathcal{M} &= \overline{N}\frac{g_A}{\sqrt{2}f_\pi}\sigma^i\tau^a q^i N\frac{i\delta^{ab}}{q^2 - M_\pi^2 + i\epsilon}\overline{N}\frac{-g_A}{\sqrt{2}f_\pi}\sigma^j\tau^b q^j N \\ &= i\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 \overline{N}\sigma^i\tau^a N\overline{N}\sigma^j\tau^a N\frac{\mathbf{q}^i\mathbf{q}^j}{|\mathbf{q}|^2 + M_\pi^2} \\ &= -iV_\pi(\mathbf{q}) \quad . \end{aligned} \tag{13}$$

This is more commonly written as

$$V_\pi(\mathbf{q}) = -\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 (\tau_1\cdot\tau_2)\frac{(\sigma_1\cdot\mathbf{q})(\sigma_2\cdot\mathbf{q})}{|\mathbf{q}|^2 + M_\pi^2} \quad , \tag{14}$$

where the subscript denotes the nucleon, either 1 or 2, and $\mathbf{q}$ is the momentum transfer in the process. As we are trying to find the static potential between two charges (the nucleon, carrying axial charge) we set $q^0 = 0$, and look at three-momentum transfer only.

The spatial potential $V_\pi(\mathbf{r})$ is recovered from the momentum space potential $V_\pi(\mathbf{q})$ via a fourier transform

$$V_\pi(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3}\; e^{i\mathbf{q}\cdot\mathbf{r}}\; V_\pi(\mathbf{q})$$

$$= -\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 (\tau_1 \cdot \tau_2) \int \frac{d^3q}{(2\pi)^3} \, e^{i\mathbf{q}\cdot\mathbf{r}} \frac{(\sigma_1 \cdot \mathbf{q})(\sigma_2 \cdot \mathbf{q})}{|\mathbf{q}|^2 + M_\pi^2}$$

$$= \left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 (\tau_1 \cdot \tau_2) \, (\sigma_1 \cdot \partial)(\sigma_2 \cdot \partial) \int \frac{d^3q}{(2\pi)^3} \, e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|^2 + M_\pi^2} \quad . \tag{15}$$

We have taken the partial derivatives outside the integral, so that it is simple to do the integration. It is clear that at $r = 0$ the integral is singular. Further, however, the integral was cubically divergent before we removed the partial derivatives, rendering a linearly divergent integral at $r = 0$. Also, not that the part of $q^i q^j$ that has the form $q^2\delta^{ij}$ gives rise to a $\delta^3(r)$ component of the potential. This has exactly the form of the four-nucleon contact terms we have not considered above. We will recover this contribution from the partial derivative acting on the result of the integral.

Let us start by doing the integral, without all the factors out the front,

$$I(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} \, e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{|\mathbf{q}|^2 + M_\pi^2}$$

$$= \frac{1}{4\pi^2} \int_0^\infty d|\mathbf{q}| \int_{-1}^1 d\cos\theta \; |\mathbf{q}|^2 \; e^{i|\mathbf{q}||\mathbf{r}|\cos\theta} \frac{1}{|\mathbf{q}|^2 + M_\pi^2}$$

$$= \frac{1}{2\pi^2 r} \int_0^\infty dq \sin(qr) \frac{q}{q^2 + M_\pi^2} \quad , \tag{16}$$

where I have set $|\mathbf{q}| = q$ and $|\mathbf{r}| = r$. It is now straight forward to do this integral by contour integration, by noting that it is an even function of q and hence the limits of integration can extended to $+ve$ and $-ve$ infinity. We then write the $\sin(qr)$ in terms of exponentials and then close in the upper and lower half-plane appropriately (with poles at $q = \pm i M_\pi$) , remember the sign arising from the direction of the contour

$$I(\mathbf{r}) = -\frac{i}{8\pi^2 r} \int_{-\infty}^{+\infty} dq \; \frac{q}{q^2 + M_\pi^2} \left[e^{iqr} - e^{-iqr}\right]$$

$$= \frac{1}{4\pi r} e^{-M_\pi r} \quad , \tag{17}$$

and therefore

$$V_\pi(r) = \frac{1}{4\pi} \left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2 (\tau_1 \cdot \tau_2) \, (\sigma_1 \cdot \partial)(\sigma_2 \cdot \partial) \; \frac{1}{r} e^{-M_\pi r} \quad . \tag{18}$$

This looks like a fairly innocent expression, but now we have to evaluate the partial derivatives acting on $I(r)$. Lets set up the partial derivatives so that it is clear where the various parts of the final expression come from, we have that

$$\partial^i\partial^j\left(\frac{e^{-mr}}{r}\right) = e^{-mr}\partial^i\partial^j\left(\frac{1}{r}\right) + \left[\partial^i e^{-mr}\partial^j\left(\frac{1}{r}\right) + i\leftrightarrow j\right] + \left(\frac{1}{r}\right)\partial^i\partial^j e^{-mr} \quad . \tag{19}$$

Now we must recall that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(r)$ and using that $\partial^j r = x^j/r = \hat{r}$ we have

$$\begin{aligned}\partial^i\partial^j\left(\frac{e^{-mr}}{r}\right) &= \left[\frac{3\hat{r}^i\hat{r}^j - \delta ij}{r^3} - \frac{4\pi}{3}\delta^{ij}\delta^3(r)\right]e^{-mr}\\ &+\left[2m\frac{\hat{r}^i\hat{r}^j}{r^2}\right]e^{-mr}\\ &+\left[m\left(\frac{\hat{r}^i\hat{r}^j}{r^2} - \frac{\delta^{ij}}{r^2} + m\frac{\hat{r}^i\hat{r}^j}{r}\right)\right]e^{-mr}\\ &= \frac{m^2}{3}\frac{e^{-mr}}{r}\left[(3\hat{r}^i\hat{r}^j - \delta^{ij})(1+\frac{3}{mr}+\frac{3}{(mr)^2}) + \delta^{ij}\right] - \frac{4\pi}{3}\delta^{ij}\delta^3(r)\end{aligned} \tag{20}$$

The final result for the potential between two nucleons induced by the exchange of a virtual pion is given by

$$\begin{aligned}V_\pi(r) &= \frac{M_\pi^2}{12\pi}\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2(\tau_1\cdot\tau_2)\left(\left[(3\hat{r}\cdot\sigma_1\hat{r}\cdot\sigma_2 - \sigma_1\cdot\sigma_2)(1+\frac{3}{mr}+\frac{3}{(mr)^2}) + \sigma_1\cdot\sigma_2\right]\frac{e^{-mr}}{r}\right.\\ &\left.-\frac{4\pi}{3}\sigma_1\cdot\sigma_2\delta^3(r)\right)\\ &= \frac{M_\pi^2}{12\pi}\left(\frac{g_A}{\sqrt{2}f_\pi}\right)^2(\tau_1\cdot\tau_2)\left(\left[S_{12}\ (1+\frac{3}{mr}+\frac{3}{(mr)^2}) + \sigma_1\cdot\sigma_2\right]\frac{e^{-mr}}{r}\right.\\ &\left.-\frac{4\pi}{3}\sigma_1\cdot\sigma_2\delta^3(r)\right) \quad .\end{aligned} \tag{21}$$

This defines the operator S_{12} as conventionally defined.

Notice that we have recovered the $\delta^3(r)$ function we discussed earlier, however, it was a bit subtle, as these things usually are! We had to add it in by hand in order to ensure that the action of the laplacian is reproduced. As we are not attempting to describes the short-distance part of the potential at this stage, and will not ask questions about operators that depend on short distance physics we can forget about the $\delta^3(r)$ contribution. This gets renormalized away into the short distance part of the potential, described by C_s and C_t, which in the language of meson exchange models correspond to ρ and ω exchange. In these models, which have many free parameters fit to reproduce nucleon-nucleon scattering, the singular part of the pion potential is "renormalized away" by the fitting of the heavy meson parameters. This fact is usually not stated in this way, but this is infact what is going on.

Let us now examine the behavour of each of the components of the pion exchange potential. The potential is strongly isospin dependent due to the $\tau_1 \cdot \tau_2 = 2\left[I(I+1) - \frac{3}{2}\right]$ factor (where I is the total isospin of the N-N system). The interactions is 3 times stronger in the $I = 0$ channel than in the $I = 1$ channel.

The contribution from the S_{12} operator depends on which angular momentum states are involved, it is a non-central force. Notice that $(3\hat{r}^i\hat{r}^j - \delta^{ij})$ is a traceless and symmetric tensor with 2 indices, and as such must transform as a $L = 2$ object under rotations. For a more pedestrian view of this consider inserting different values for $\hat{r}$. For $i = j = 3$ we have $\hat{r} = \hat{z} = \cos\theta$, and $(3\hat{r}^3\hat{r}^3 - \delta^{33}) = 3\cos^2\theta - 1 = 2\frac{\sqrt{4\pi}}{5}\ Y_2^0(\theta\phi)$ or $i = 3, j = 1$ gives $(3\hat{r}^3\hat{r}^1 - \delta^{31}) = 3\cos\theta\sin\theta\cos\phi = -\sqrt{\frac{2\pi}{15}}\,(Y_2^1 + Y_2^{-1})$. Both of which are written entirely in terms of the Y_2^m tensors. Therefore, this term can only contribute when the angular momentum of the initial and final states differs by $\Delta L = 2$. So for, initial and final state S-waves, it doesn't contribute. However, in the case of the deuteron, where we have a total $J = 1$ state, this operator can induce and admixture of $L = 2$ into our initially $L = 0$ state. It is now clear that we must consider both components from the beginning and perform a coupled channels analysis. Physically, the form of this potential indicates that it is strongest when the separation vector $\mathbf{r}$ is aligned or antialigned in the direction of the spins. The sign of the contribution depends upon the isospin channel under consideration, but for the deuteron with isospin $I = 0$, the noncentral force deforms the deuteron to be cigar-shaped. We should note that this form of interaction preserves spin, i.e. it commutes with the S^2 operator. It is this interaction that is responsible for the quadrupole moment of the deuteron.

The central part of the potential induced by pion exchange (preserves angular momentum by definition) depends on the spin and isospin channel.

It is attractive for $I = 0, S = 1$, $I = 1, S = 0$ but repulsive for the $I = 0, S = 0$ and $I = 1, S = 1$ channels, to name a few.

References