PHYS 560: Lectures During Week 2

Martin J. Savage

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Abstract

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1 Scattering in a Central Potential

I hope that most of this is just a reminder to you of work covered in other classes, hopefully the introductory quantum lectures. We are considering the situation of a wavepacket, localized at -ve ∞ at time $t = -\infty$. We wish to form the (differential) cross section for this wave packet to scatter off a scattering centre. This amounts to determining the probability for the particle in the wave packet to end up travelling at some angle with respect to its initial direction. Wave packets are inherently messy objects to deal with and so it is more convenient to determine the scattering amplitudes of incident plane waves, and the convolve with the wavepacket amplitudes at the end. In reality we never do this as the momentum spread of the wavepacket, is small compared to other scales in the problem.

When we are far from the scattering potential for a finite range potential, then we know that the incident wavefunction is just a plane wave e^{ikz}, where $k = |\mathbf{k}| = \sqrt{2\mu E}$ where μ is the reduced mass of the system. As we are going to be dealing with scattering in states of angular momentum l it is convenient to expand the incident plane wave in the z-direction into angular momentum states

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0} (2l+1)\ i^l\ j_l(kr)\ P_l(\cos\theta) \quad . \tag{1}$$

It is obvious that this plane wave is a solution to the schrodinger equation with no potential. In order to determine the amplitude for scattering of this incident plane wave we need to solve the schrodinger equation with the complete potential $V(r)$ (we are assuming a central potential). We have that

$$\begin{aligned} H\psi &= E\psi \\ \left[-\frac{1}{2\mu}\nabla^2 + V(r)\right]\psi(\mathbf{r}) &= -E\psi(\mathbf{r}) \quad . \end{aligned} \tag{2}$$

We use the separability of ψ and define

$$\psi(\mathbf{r}) = \frac{1}{r}u(r)_l Y_l^m(\Omega) \quad , \tag{3}$$

to write

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + 2\mu V(r) - k^2\right] u_l(r) = 0 \quad , \tag{4}$$

where we have used that the radial part of the laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \text{angular.....} \quad . \tag{5}$$

We have a boundary condition on $u_l(r)$ arising from the fact that the wavefunction ψ must be finite everywhere. This gives that u must vanish at the origin, $u_l(0) = 0$. The slope of u_l at the origin is determined by the normalization of the wavefunction, and can be chosen to be a real number, making u_l a real valued function.

If there was no potential $V(r) = 0$ everywhere, then the solution to the schrodinger equation would be $u_l(r) \sim r j_l(kr)$, where $j_l(z)$ is a spherical Bessel function. However, when there is a non-zero potential, we can still have u_l vanish at the origin, while at long distances (much larger than the range of the potential) the solution has the form

$$\begin{aligned} u_l(r) &= r\left(A_l j_l(kr) + B_l n_l(kr)\right) \\ &\to \frac{1}{k}\left(A_l \sin(kr - l\frac{\pi}{2}) - B_l \cos(kr - l\frac{\pi}{2})\right) \quad , \end{aligned} \tag{6}$$

as $r \to \infty$. The eigenfunctions are $j_l(z)$ (the spherical Bessel functions) which are regular at the origin, and $n_l(z)$ (the Neumann functions) which are singular at the origin. Our statements about boundary conditions lead us to $B_l = 0$ when $V = 0$.

It is useful to have the asymptotic forms of the Bessel and Neumann functions explicitly to understand the behaviour of the amplitudes

$$\begin{aligned} j_0(z) &\to \frac{\sin z}{z} \\ j_1(z) &\to \frac{\sin z}{z^2} - \frac{\cos z}{z} \\ j_2(z) &\to \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \\ n_0(z) &\to -\frac{\cos z}{z} \\ n_1(z) &\to -\frac{\cos z}{z^2} - \frac{\sin z}{z} \\ n_2(z) &\to -\left(\frac{3}{z^3} - \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z \quad . \end{aligned} \tag{7}$$

For a non-vanishing potential, we are free to define the ratio of coefficients A_l and B_l to be

$$\frac{B_l}{A_l} = -\tan\delta_l \quad , \tag{8}$$

where δ_l is a energy dependent phase. Further, for convenience we normalize the states to have a coefficient of unity,

$$u_l(r) \rightarrow \frac{1}{k}\sin(kr - l\frac{\pi}{2} + \delta_l) \quad . \tag{9}$$

The complete wavefunction (denoted by ψ^+) for the scattering of a plane wave is the sum of the incident plane wave and the outgoing wave (outgoing means e^{ikr} and incoming means e^{-ikr}) is

$$\begin{aligned} \psi^+ &= e^{ikr\cos\theta} + f(\theta)\frac{e^{ikr}}{r} \\ &= \sum_{l=0}(2l+1)i^l\frac{\sin(kr - l\frac{\pi}{2})}{kr}P_l(\cos\theta) + f(\theta)\frac{e^{ikr}}{r} \quad , \end{aligned} \tag{10}$$

and we know that this can be written in terms of eigenstates of the hamiltonian, including the potential,

$$\begin{aligned} \psi^+ &= \sum_{l=0} a_l P_l(\cos\theta)\frac{u_l}{r} \\ &\rightarrow \sum_{l=0} a_l P_l(\cos\theta)\frac{\sin(kr - l\frac{\pi}{2} + \delta_l)}{kr} \quad , \end{aligned} \tag{11}$$

for unknown coefficients a_l. By comparing the asymptotic form of the incoming waves (e^{-ikr}) of these expressions, we find that the a_l's are $a_l = (2l+1)i^l e^{+i\delta_l}$ leading to

$$\psi^+ = \sum_{l=0}(2l+1)i^l e^{i\delta_l}P_l(\cos\theta)\frac{\sin(kr - l\frac{\pi}{2} + \delta_l)}{kr} \quad . \tag{12}$$

A comparison now, between the outgoing waves (e^{+ikr}) yields

$$\begin{aligned} f(\cos\theta) &= \frac{1}{2ik}\sum_{l=0}(2l+1)(e^{2i\delta_l} - 1)P_l(\cos\theta) \\ &= \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l \; P_l(\cos\theta) \quad . \end{aligned} \tag{13}$$

We are now in a position to determine the cross section for a given process. We start by computing the flux of particles associated with ψ^+. From NR quantum mechanics we have that the current density is

$$\mathbf{j} \;=\; \frac{1}{\mu}\mathrm{Re}\,(\psi^*\nabla\psi) \quad , \tag{14}$$

from which we have that

$$\begin{aligned} \mathbf{j}_{\mathrm{inc}} \;&=\; v \;=\; \frac{k}{\mu} \\ \mathbf{j}_{\mathrm{scatt}} \;&=\; v\frac{1}{r^2}|f(\Omega)|^2 \quad . \end{aligned} \tag{15}$$

The differential cross section is defined to be

$$\frac{d\sigma}{d\Omega} \;=\; |f(\Omega)|^2 \quad , \tag{16}$$

and the probability of scattering from a beam incident upon a target with N scatterers per unit volume and thickness T into a solid angle $d\Omega$ is

$$\mathrm{prob} \;=\; \frac{d\sigma}{d\Omega}d\Omega NT \quad , \tag{17}$$

assuming that the beam is negligibly attentuated in passing through the target.

Hence the cross section is

$$\begin{aligned} &\frac{d\sigma}{d\Omega} = |f(\cos\theta)|^2 \\ &= \frac{1}{k^2}|\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)|^2 \\ &\sigma = \frac{4\pi}{k^2}\sum_{l=0}(2l+1)\sin^2\delta_l \quad , \end{aligned} \tag{18}$$

where we have used the orthogonality of Legendre polynomials

$$\int dx\ P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'} \quad . \tag{19}$$

We see from this that the unitarity of the scattering amplitude ($|\sin\delta_l| \leq 1$) leads to the contribution from each partial wave being bounded from above by

$$\sigma_l \leq \frac{4\pi}{k^2}(2l+1) \quad . \tag{20}$$

2 Low-Energy S-Wave Scattering

Let us familiarize ourselves with these scattering amplitudes by examining low energy scattering in the S-wave. In this scenario we have only one partial wave contributing (obviously) with a phase shift δ_0. The cross section for S-wave scattering is then given in terms of this one phase shift

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 \quad . \tag{21}$$

It is clear that for a finite range potential, such as the strong interaction potential between nucleons, the cross section is finite for all energies, and hence the phase shift δ_0 must vanish at least as fast as k in order, so that $\sin\delta_0/k$ is finite. Hence for small k the phase shift will behave as $\delta_0 \rightarrow -ka$, where a is some constant with dimensions of length (to compensate the dimensions of mass of k). In this limit one can also replace the $\sin\delta_0$ with δ_0, to yield a cross section in terms of a

$$\sigma = 4\pi a^2 \quad . \tag{22}$$

The constant a is called the *scattering length.*

In the limit of $k \rightarrow 0$ as we have been considering above, the solution to the schrodinger equation in the region where the potential vanishes

$$-\frac{1}{2\mu}\nabla^2 = 0 \quad , \tag{23}$$

is a straight line. We already know this as the solution for any k is

$$\begin{aligned} u_0 &\rightarrow \frac{\sin(kr+\delta_0)}{k} \\ &\rightarrow r - a \quad , \end{aligned} \tag{24}$$

in the low momentum regime. We see that the scattering length a is the radius at which the asymptotic (straight line) solution vanishes. This is, of course, a limit of the incident oscillatory waves.

One of the things we should learn from this is that all we see in the low energy (compared to the inverse radius of the potential) is the asymptotic behaviour of the wavefunction characterized by one constant, the scattering length. This is a nontrivial construction from the potential itself and the wavefunction of the state. We see that for momenta much less than the inverse radius of the potential the scattering length is sufficient to describe all of the interactions. This goes back to our original ideas about short distance physics being encoded in the constants of the low energy theory. It is clear that by measuring the scattering length of a system alone we cannot reconstruct the potential uniquely. There are infinitely many different shapes, depths and ranges of potentials that will reproduce a single scattering length.

Let us consider the system I have shown in the which corresponds to a

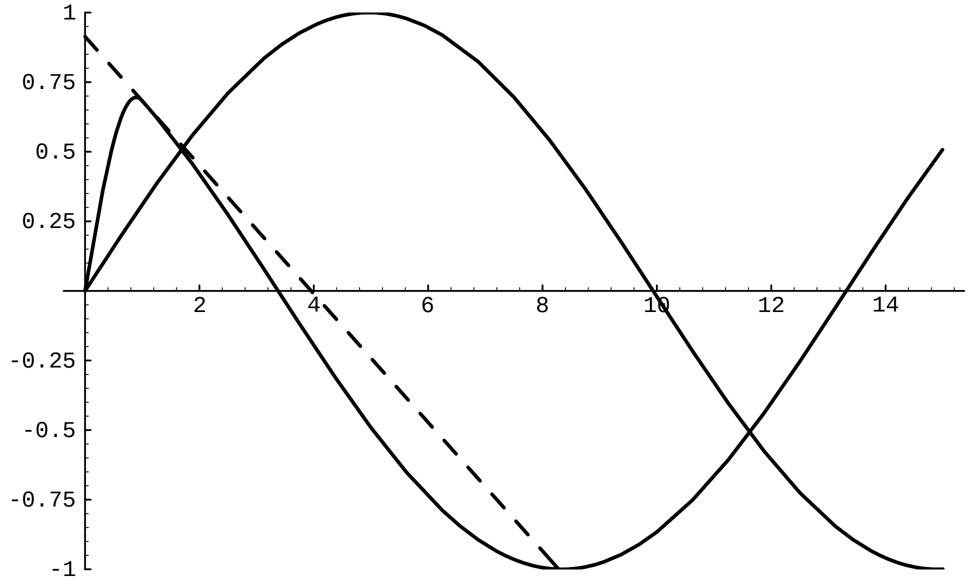

Figure 1: *The wavefunctions at low momentum for the potential in eq. (25).*

finite square well

$$\begin{aligned} V(r) &= -3 \quad \text{for } r < 1 \\ &= 0 \quad \text{for } r > 1 \end{aligned} \quad , \tag{25}$$

in which a system of reduced mass $\mu = 1/2$ moves. The oscillatory curves correspond to the wavefunctions for particles incident upon the potential of energy $E = 0.1 = k^2$ for $\mu = 1/2$. One is for the system with the potential given above while the other is that for a vanishing potential. I have also shown the $E = 0$ wavefunction for the potential. We see that the oscillatory wavefunction intercept with the x-axis is tending to the scattering length

$r = a$ defined by the $E = 0$ intercept, shown by the straight line for $r > 1$. The phase shift (divided by k) corresponds to the difference between the intercepts with the x-axis, i.e. the difference between $a \sim 4$ and $r_1 \sim 10$. This would say that the scattering amplitude is given by

$$\begin{aligned}
\text{amp} &\sim \frac{\sin\delta}{k} \\
&= \frac{\sin(k(r_1 - a))}{k} \\
&= \frac{\sin(\pi - k(r_1 - a))}{k} \\
&= \frac{\sin(ka)}{k} \\
&\to a \quad ,
\end{aligned} \tag{26}$$

where we used the fact that $kr_1 = \pi$ by construction.

It is important to notice that for $a > 0$ the wavefunction inside this potential must have just turned over to have a negative derivative at the boundary. This would indicate that a bound state exists for this potential. In order for us to arrive at this conclusion we see that the existence of a bound state requires that the wavefunction at the boundary have a negative derivative so that we can match onto an exponentially falling wavefunction outside the potential. Such a fall off is characteristic of a state with negative energy $E < 0$, and is localized about the centre of the potential. We will return to this later on. Also, we note that if the scattering length were negative $a < 0$, then the wavefunction would have a positive derivative at the boundary of the potential. This indicates that there is not a loosely bound ($E <\sim 0$) state for this system.

So far we have considered $k = 0$ scattering and found that a single parameter can be identified, the scattering length, and we have discussed what this means in terms of wavefunctions and cross sections. However, when we move away from $k = 0$ scattering and ask about the cross section for non-zero momenta but still much smaller than the inverse radius, then we expect to be able to perform a taylor expansion on the scattering amplitude, or some quantity related to the scattering amplitude. We have observed in our previous discussions that we can "encode" (in some complicated way) the potential and wavefunction in this region by the phase shift of the eigenstates of the potential. We expect that the dependence on small momenta (compared with the potential range) can also be encoded by this phase shift.

We therefore wish to form a derivative expansion of the phase shift, or some function of the phase shift, if it is more appropriate. Let $\chi(r)$ be the asymptotic form of $u_0(r)$ as $r \to \infty$, where the radial wavefunction is $\psi = u/r$. We have already seen that $\chi(r) = A\sin(kr+\delta)$ and we can form the quantity

$$\frac{1}{\chi(0)}\left[\frac{\partial\chi(r)}{\partial r}\right]_{r=0} = k\cot\delta \quad . \tag{27}$$

We choose to perform the taylor expansion of this object in k^2. It is quadratic in k because we are perturbing the solution to the schrodinger equation about $k=0$, and k appears quadratically. We define the *effective range* $r_{\rm eff}$ by

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r_{\rm eff} k^2 + ... \quad , \tag{28}$$

where the $k \to 0$ limit coincides with our previous definition of the scattering length.

Before we proceed, lets look at this object a slightly different way. Starting with the scattering amplitude $f(\Omega)$ for an S-wave interaction

$$\begin{aligned} f(\Omega) &= \frac{1}{k} e^{i\delta}\sin\delta = \frac{1}{k}\frac{\sin\delta}{\cos\delta - i\sin\delta} \\ &= \frac{1}{k\cot\delta - ik} \quad . \end{aligned} \tag{29}$$

Therefore, if we understand the object $k\cot\delta$, then we understand the scattering amplitude, and hence the cross section. Many years ago, Schwinger and others examined the properties of $k\cot\delta$. For a finite range potential, range $\sim R$, the quantity $k\cot\delta$ has a power series expansion about $k = 0$ that converges for $k < 1/R$. We write

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + r_1 k^4 + ... \tag{30}$$

It is easy to see that this correctly reproduces the small k behaviour of δ. In fact one can show, as may already be obvious from the figure that the scatterinbg length is not bounded by the range of the potential, but the r_i are set by the range of the potential, i.e.

$$-\infty < a < \infty \quad , \quad r_i \sim R \quad . \tag{31}$$

We will spend some time discussing where the following numbers come from, but it is found that from neutron proton scattering

$$\begin{aligned} a^{(^1S_0)} &= -23.714 \pm 0.013 \text{ fm} \quad , \quad r^{(^1S_0)} = 2.73 \pm 0.03 \text{fm} \\ a^{(^3S_1)} &= 5.425 \pm 0.0014 \text{ fm} \quad , \quad r^{(^3S_1)} = 1.749 \pm 0.008 \text{fm} \end{aligned} \tag{32}$$

An interesting feature of $k \cot\delta$, is that when the entire denominator of $f(\Omega)$ vanishes, the scattering amplitude becomes infinite, corresponding to the existance of a bound state at that location. Therefore, solving $k\cot\delta - ik = 0$ determines the location in k space of bound states in either channel. In the $^3S_1$ channel, this corresponds to the deuteron bound state with a binding energy of $B =\sim 2.2$ MeV. In the $^1S_0$ channel the pole has positive energy corresponding to a resonance in the scattering amplitude. In fact, the location of the deuteron pole is one of the inputs into the precise determination of $a^{(^3S_1)}$ and $r_0^{(^3S_1)}$.

3 Zero Energy n-p scattering

It is easiest to think about the scattering of a n and a p in terms of the scattering of partial waves denoted by spin and isospin (we have set $l = 0$) as we are at vanishing incident energy. The total wavefunction must be antisymmetric as we are dealing with fermions, and hence the scattering states must have either $I = 0, S = 1$ or $I = 1, S = 0$. (We are, of course, writing it in such a way to easily extend to nn and pp scattering. These processes can only occur in the $I = 1, S = 0$ for S-wave scattering as they must be $I = 1$ only.) Each of these scattering channels will have an associated scattering length, effective range and any other parameter defined to describe the energy dependence of the scattering amplitude. By conservation of angular momentum, the $L = 0, S = 0$ and $L = 0, S = 1$ states do not mix under the scattering interaction. An example of where the channels do mix would be at non-zero incident energy, we would have the possibility of a $L = 2, S = 1$ and a $L = 0, S = 1$ state, both with $J = 1$, mixing with each other. This would require a coupled channels analysis. Let us ignore this type of problem to begin with.

For an unpolarized beam incident upon an unpolarized target we can simply determine the relative contribution of the $S = 1$ channel to $S = 0$ channel to be 3 to 1, simply by the number of substates. The spin-averaged cross section is given in terms of the $S=0$ (spin-singlet) and $S=1$ (spin triplet) scattering lengths a_1 and a_0, respectively, is then

$$\sigma(E=0) = 4\pi\left(\frac{3}{4}a_1^2 + \frac{1}{4}a_0^2\right) \quad . \tag{33}$$

The spin averaging gives the factor of $\frac{1}{4}$ from the four different spin configurations of 2 spin $\frac{1}{2}$ particles.

Let us go back on step and write the amplitude for a given process in terms of the triplet and singlet scattering amplitudes. The cross section in terms of the scattering length is

$$\sigma = 4\pi a^2 \quad , \tag{34}$$

which lead to eq. (33) after performing spin averaging over the two spin channels available to np S-wave interactions. It is convenient to write the amplitude a in terms of a spin operator that acts on the particles involved. This saves us from defining the amplitude for the $S=1$ channel and the amplitude for the $S=0$ channel separately. We know that for $S=1$ the amplitude is $a=a_1$ and for $S=0$ the amplitude is $a=a_0$. Now $S=s_n+s_p$ and $S^2=(s_n+s_p)^2$ and hence

$$\begin{aligned} s_n\cdot s_p &= \frac{1}{2}\left[S(S+1)-s_n(s_n+1)-s_p(s_p+1)\right] \\ &= \frac{1}{4} \quad \text{for} \quad S=1 \\ &= -\frac{3}{4} \quad \text{for} \quad S=0 \quad . \end{aligned} \tag{35}$$

It follows that the amplitude given in terms of the above spin operator is

$$a = \frac{1}{4}\left(3a_1+a_0\right) + \left(a_1-a_0\right) s_n\cdot s_p \quad . \tag{36}$$

So far we haven't really gained anything by rewriting the amplitude in an operator language, but in fact we shall see that this allows us to extend the analysis to systems composed of multiple nucleons. In particular, consider scattering neutrons from molecular hydrogen, H_2. If the wavelengths of the incident neutrons are short compared to the inter-atomic spacing between the two protons in H_2, then molecule is likely to be blown apart when interacting with the incident neutron. However, if the incident energy and hence inverse wavelength is much less than the inter-atomic spacing ($\ll 7.8 \times 10^{-11} m$, in fact it is required that $E^{-1} \ll 2 \times 10^{-10} m$) then the neutron will only be able to elastically scatter from the H_2 molecule and hence the amplitudes for scattering from the two protons will add coherently. The operator giving rise to the amplitude (denoted by a_H) for scattering from H_2 is the sum of eq. (36) over the two protons, giving

$$a = \frac{1}{2}\left(3a_1 + a_0\right) + \left(a_1 - a_0\right) s_n \cdot s_H \quad , \tag{37}$$

where S_H is now the spin of the H_2 molecule, either $S_H = 0$ for para-hydrogen or $S_H = 1$ for ortho-hydrogen.

Now to determine the scattering amplitudes. For para-hydrogen ($S = 0$) we have that $s_n \cdot s_H = 0$, which leads to $a_{\mathrm{para}} = \frac{1}{2}\left(3a_1 + a_0\right)$ and hence a cross section of $\sigma_{\mathrm{para}} = \pi\left(3a_1 + a_0\right)^2$, after spin-averaging and summing over the number of substates (in this case 2 corresponding to the two spin states of the neutron). It is a bit trickier for ortho-hydrogen, and most texts have entirely unsatisfactory descriptions in my opinion. When we have $s_H = 1$, then the $n - H_2$ system can have total $S = 3/2$ or $1/2$ and we must perform the appropriate spin averaging over the summed matrix element. There are a total of $2 \times 3 = 6$ spin channels available 4 in the $S = 3/2$ channel and 2 in the $S = 1/2$ channel. Therefore the cross section for n scattering off ortho-hydrogen is

$$\sigma = 4\pi\frac{1}{6}\left[4|a_{S=3/2}|^2 + 2|a_{S=1/2}|^2\right] \quad , \tag{38}$$

where is is easy to show that

$$\begin{aligned} a_{S=3/2} &= 2a_1 \\ a_{S=1/2} &= \frac{1}{2}a_1 + \frac{3}{2}a_0 \quad , \end{aligned} \tag{39}$$

where we have used $s_n \cdot s_H = \frac{1}{2}\left[S^2 - s_n^2 - s_H^2\right]$. One finds that eq. (39) can be rewritten as

$$\sigma = \pi(3a_1 + a_0)^2 + 2\pi(a_1 - a_0)^2 \quad . \tag{40}$$

The total cross section for zero-energy np scattering and the coherent amplitude for scattering long-wavelength neutrons from molecular hydrogen have been measured very well and it is found that

$$\begin{aligned} \sigma(0) &= 20.442 \pm 0.23\ b \\ \frac{1}{2}\left(3a_1 + a_0\right) &= -3.707 \pm 0.008\ fm \quad , \end{aligned} \tag{41}$$

where the "barn" is defined by

$$1b = 10^{-24}\mathrm{cm}^2 = 100\mathrm{fm}^2 \quad . \tag{42}$$

We have two unknowns and two independent measurements from which we can determine that

$$\begin{aligned} a_0 &= -23.710 \pm 0.030\ \mathrm{fm} \\ a_1 &= 5.432 \pm 0.005\ \mathrm{fm} \quad . \end{aligned} \tag{43}$$

There is a lot of physics in these two numbers. Firstly, notices that they are both large, larger than one would have guessed from the naive length scale associated with nuclear interactions, ~ 1fm. We learned last time that the scattering length is not really a measure of the range of the potential, but tells us more about the behaviour of the wavefunction, it is where the $E = 0$ wavefunction intercepts the $\psi = 0$ axis. Further, we saw that it is not a bad thing to have a large, or even infinite scattering length, it merely means that the $E = 0$ wavefunction is a straight horizontal line, resulting from the boundary condition of the wavefunction emerging from the region of the potential. The case at hand, where both the spin singlet and triplet channels have large scattering lengths indicates that the nuclear force is somewhat insensitive to spin-dependent interactions (we will return to this later). It is clear that there is some spin dependence to the interaction, but it is weak. Such scattering lengths can be understood if in the spin triplet channel there is a weakly bound state, so that the wavefunction just turns over at the boundary of the potential giving a large but $+ve$ scattering length. Similarly, if there is a barely, unbound state in the spin-singlet channel, so that their wavefunction just fails to turn over at the edge of the potential, giving a large but negative scattering length.

Further, we know that in the $S = 1$ channel we have the deuteron. Neglecting, for the moment the $D - wave$ admixture in its wavefunction we can use the formula relating the binding energy to the scattering length and effective range to find that

$$r_{\mathrm{eff}}^{(1)} = 1.749 \pm 0.008\ \mathrm{fm} \quad . \tag{44}$$

This information then allows us to use low energy scattering to determine the effective range in the $S = 0$ channel, we find that

$$r_{\rm eff}^{(0)} = 2.73 \pm 0.03 \ {\rm fm} \quad . \tag{45}$$

References