PHYS 560: Lectures During Week 1

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October 4, 1999

Abstract

The first week of lectures for **Nuclear Physics**, PHY560, Autumn 1999.

September 1999.

1 Course Structure

Lets start by discussing the structure of the course, PHY560. This course is an introduction to nuclear physics, with graduate level quantum mechanics as a prerequisite. The text for the course is **Introductor Nuclear Physics** by **Samuel S. M. Wong**, and can be purchased at the UW bookstore. You will find it very helpful to have other texts, beside this one to refer to during the quarter. On the PHY560 web page (which is linked into the Physics Dept. homepage, and also to my home page) I have listed a few reasonably good books, and I may add more as time goes by. There will be assignments distributed every week that I expect you to complete and turn into me the following week for grading. The course grade will be determined by these assignments.

2 Introduction to Nuclear Physics

Nuclear Physics is physics at lengths scales around 1 fm, and its study started around the turn of the century with several important experimental discoveries. As you all know, it was the inadequacy of J.J. Thompson's "Plum-Pudding Model" of the atom to describe the back-angle scattering of α-particles from atoms that lead Ernest Rutherford (a New Zealander) to propose the "Solar System" (if you like) model of the atom. In Rutherford's model, most of the mass of the atom, of radius $r_{\rm atom} \sim 10^{-8}$ cm. is concentrated in a much smaller volume of radius, $r_{\rm nucleus} \sim 10^{-13}$ cm, leaving most of the atom void from mass except that carried by the atomic electrons. This is in contrast to Thompson's model where the mass is uniformly distributed over the atomic volume, with the electrons stuck in like the plums in a plum-pudding. Rutherford's model, one with a compact hard scattering core, could explain a large cross section for back-angle α-scattering, or as we know it know, "Rutherford Scattering". He was looking at electromagnetic scattering (Nuclear Interactions were not known about at the time) and hence was guided by what he knew about electromagnetic interactions. Previous models had the charge distribution smeared out over a large volume, whereas he found that the charge was required to be highly localized, point-like, on the scale of his probe. This type of investigation continued for the rest of this century. Large angle scattering which is physics of today, becomes the small-angle backgrounds of the physics of tomorrow. Rutherford scattering is a prime demonstration of this. As one fires in smaller and smaller wavelength objects to probe the response of a given system, at some energy the wavelength becomes short enough to see the finite extent of the scattering center and hence things start looking like the Thompson model again! On should also recall that at Rutherford's time, and for many years after, the neutron was not known about, the only "fundamental" particles were the electron and the proton. It was only in 1932, that Chadwick discovered the neutron. This really was the start of nuclear physics as we know it (in my opinion of course).

The phenomenology of nuclear physics is vast, by which I mean that there are many very interesting effects, observations and descriptions that exist. The theory at a descriptive level (e.g. the shell model) is well developed, and describes the features of nuclei and their excitations at an acceptable level. Understanding nuclear physics at a fundamental level (i.e. from QCD) is still in its infancy. The hope is that the global symmetries of QCD (e.g. chiral symmetry) will constrain hadronic interactions sufficiently to give rise to calculable nuclear observables. Such ideas are not new, but they are difficult to implement. It is only over the last few years that such a program has become feasible and that nuclear physicists have become interested. Significant progress has been made in the one-, two- and three-body sectors by employing effective field theory techniques. I hope that we will be able to discuss these developments during the next quarter. If you have had a course in field theory you might be interested in looking at the recent work (e.g. [1] and references therein).

It seems logical that the approach to take is to understand the one-nucleon, two-nucleon and other few nucleon systems at all relevant momentum transfer and then go ahead and compute nuclear observables, such as binding energies, magnetic moments, quadrupole moments and anything else you might think of doing. Certainly, in my mind this is the only approach that ultimately allows us to understand what is going on. However, the real world is not so nice, but just as pretty! Collective effects play a central role in much of nuclear physics. For example, nuclei in the $A = 20$ region are highly "deformed", which can be seen by the relative size of electric quadrupole radiation emitted in transitions between excited states. Such collective effects are difficult to see when we are only looking at the microscopic interactions. Hence, it is sometimes much easier to go to a much more phenomenological approach. The shell-model is a good example of this where the nucleons are (to first approx) non-interacting objects moving in independent single particle orbits of a central potential, whose radial dependence is determined empirically. Such models works OK. We will look at this in much more detail later.

3 A General Picture of QCD

The underlying theory of strong interactions is Quantum Chromodynamics (QCD), which is responsible for all strong interaction phenomena. The QCD lagrange density is

$$\mathcal{L}_{\mathrm{QCD}} = \overline{q}_i^a \left[i\not{D} - m_i \right]_a^b q_b^i - \frac{1}{4} G_{\mu\nu}^A G^{A\ \mu\nu} \quad , \tag{1}$$

where $q_i^a(x)$ is a quark field operator, where $i = u, d, s, c, b, t$ are the six standard model quarks with mass m_i and a is a colour index that runs from $a = 1, 2, 3$ (or red, green, blue depending on your taste), e.g.

$$u^a = \begin{pmatrix} u \\ u \\ u \end{pmatrix} \quad . \tag{2}$$

The covariant derivative that preserves local invariance under SU(3) color transformations is $\not{D} = \gamma_\mu D^\mu$ where $D_\mu = \partial_\mu + i g_s T^A A_\mu^A$, where A_μ^A is a gluon field, $A = 1, .., 8$. The SU(3) field strength tensor is defined through $\frac{1}{i g_s} [D_\mu, D_\nu] = G_{\mu\nu} = G_{\mu\nu}^A T^A$, where T^A are the SU(3) Gell-Mann matrices, and g_s is the strong interaction coupling "constant".

This looks like a very simple theory, there are quarks q_i^a which interact with each other via gluons A_μ^A. This part looks exactly like electromagnetism, however there is a beast hidden in $G_{\mu\nu}^A G^{A\ \mu\nu}$, and that is the non-linear interaction between gluons. Unlike photons which do not interact with themselves (Maxwells equations are linear), the eight gluons can interact with each other through the three and four gluon interactions, a generic feature of a non-abelian theory (Yang-Mills theory). Through quantum fluctuations these interactions make the strong coupling g_s depend upon the scale at which the particular process is occurring, i.e. it is not constant. Even more interesting is that g_s gets larger for smaller momentum transfers (long-distances) and weaker for high momentum transfers (short-distances). This is called asymptotic freedom. Numerous experiments have been performed over the last few years and theoretical techniques developed to determine the value of g_s in the region where it is small. I have "borrowed" (without permission from the ALEPH Collaboration website[2]) a plot of the coupling as a function of the "scale" at which the process is occurring. In fig. (1) the quantity $\alpha_s = g_s^2/(4\pi)$ is plotted. Eventually, as the momentum-scale becomes

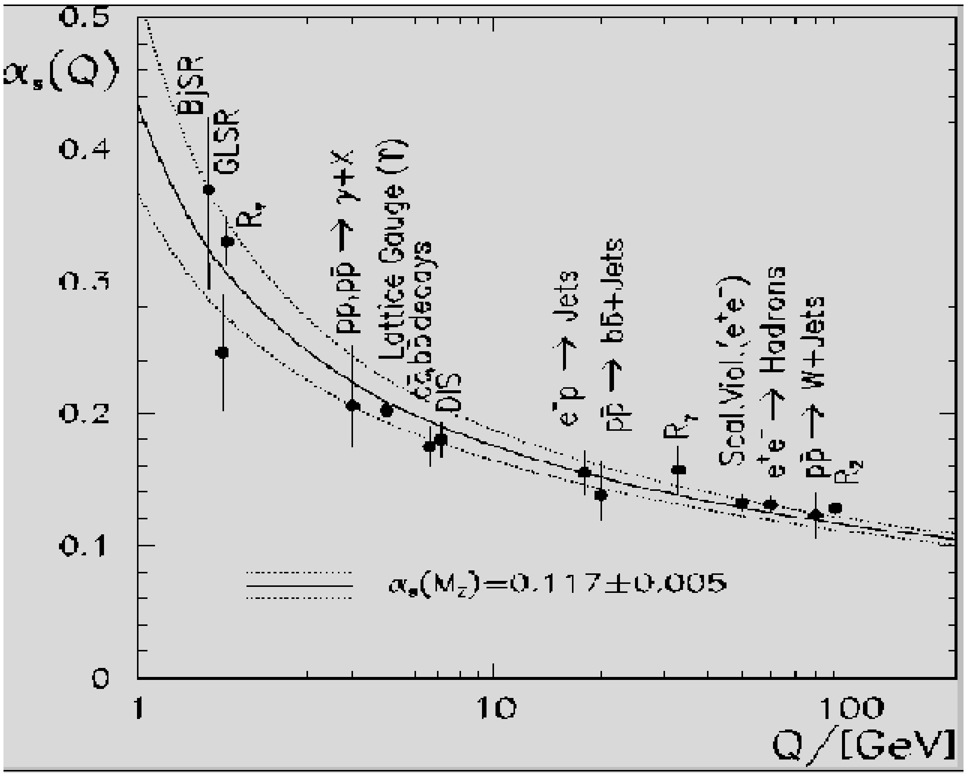

Figure 1: *The strong interaction coupling constant $\alpha_s(Q^2)$ as a function of Q^2.*

smaller, i.e. we look at processes involving larger and larger distance scales, the coupling constant becomes large, too large for a perturbative treatment. The theory becomes non-perturbative and in this regime the quarks and gluons are no-longer perturbatively close to what the "in" and "out" states of a particular interaction process look like. In fact, no-one has ever observed an isolated quark or gluon....there is no evidence that free quarks or gluons exist. In fact the interaction between them become so strong that they ALWAYS bind into hadrons, such as protons, neutrons, pions, Δ's and so forth. These are states that are singlet under SU(3) colour transformations (it is worth spending a few minutes looking at the particle data book to see what we know about the observable hadrons). Thus, if we are interested in the low-momentum interactions of, say, nucleons then it is clear that we cannot use a perturbative approach in terms of quark and gluon fields and we have to develop a theory that is written entirely in terms of hadronic fields. A somewhat more technical description goes along the lines of the approximate $SU(2)_L \otimes SU(2)_R$ chiral symmetry of the lagrange density in eq. (1) is spontaneously broken to $SU(2)_V$, by an non-zerop value of $\langle 0|\overline{q}q|0\rangle$. What all of this means will be covered in the next quarter. During this quarter we will study the phenomenology of the hadronic theory and look for "exact" results basd on the symmetries of QCD where they exist.

A feature that is obvious but one that we have not stressed is that the strong interactions between quarks and gluons doe NOT depend upon the flavor of the quark. The coupling between gluons and the u-quark is the same as the coupling between gluons and the d-quark. The two differences between the quarks are firstly their masses, and secondly their electric charges.

3.0.1 Chiral Symmetry and Isospin Symmetry

Let us examine the part of $\mathcal{L}_{\rm QCD}$ that describes the u (up), d (down) quarks and gluons only in the limit of $m_{u,d} = 0$,

$$\begin{aligned} \mathcal{L}_{\rm QCD} &\rightarrow \overline{Q} i \not{D} Q - \frac{1}{4} G^A_{\mu\nu} G^{A\ \mu\nu} \\ &= \overline{Q}_L i \not{D} Q_L + \overline{Q}_R i \not{D} Q_R - \frac{1}{4} G^A_{\mu\nu} G^{A\ \mu\nu} \\ Q_L &= \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad , \quad Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \quad . \end{aligned} \tag{3}$$

This lagrange density is clearly invariant under the global transformations

$$Q_L \rightarrow L Q_L \quad , \quad Q_R \rightarrow R Q_R \quad , \tag{4}$$

where L and R are independent SU(2) unitary transformation matrices, $L^\dagger L = R^\dagger R = 1$. If we choose $L = R = V$, this corresponds to transforming the left and right handed fields by the same "amount". If the world "looked" the same no matter what we choose for V then we would say that the world was invariant under isospin transformations (I will use V to denote the transformation in this section. Be aware that I will change notation in the near future for ease). In fact, isospin is found to be a very good symmetry of nature, that is to say that the world only changes a little bit under isospin transformations.

It turns out that when we consider the non-zero values of the quark masses $m_u \sim 5$ MeV and $m_d \sim 10$ MeV, isospin symmetry is explicitly broken,

$$\begin{aligned} \overline{q}_L \gamma_\mu q_L \rightarrow \overline{q}_L L^\dagger \gamma_\mu L q_L &= \overline{q}_L \gamma_\mu q_L \\ \overline{q}_R \gamma_\mu q_R \rightarrow \overline{q}_R R^\dagger \gamma_\mu R q_R &= \overline{q}_R \gamma_\mu q_R \\ \overline{q}_R m_Q q_L \rightarrow \overline{q}_R R^\dagger m_Q L q_L &\neq \overline{q}_R m_Q q_L \quad , \end{aligned} \tag{5}$$

with $m_Q = diag(m_u, m_d)$. Given that the u and d masses differ by a factor of two it might first appear that isospin will be badly broken. However this is not the case. The typical scale for the size of hadronic interaction strengths is set by the scale at which spontaneous symmetry breaking occurs, i.e. when

$$SU(2)_L \ \otimes \ S(2)_R \ \rightarrow \ SU(2)_V \quad , \tag{6}$$

due to g_s becoming large. This scale is called the chiral symmetry breaking scale and is

$$\Lambda_\chi \ \sim \ 1 \text{ GeV} \quad . \tag{7}$$

Therefore, we conclude that the appropriate small dimensionless parameter that describes isospin breaking is

$$(m_d - m_u) / \Lambda_\chi \sim 5 \times 10^{-3} \quad . \tag{8}$$

Thus, isospin symmetry is an accidental symmetry of nature. It is an accident because the scale of chiral symmetry breaking and the mechanism that produces the quarks masses (the Higgs mechanism) are unrelated.

4 Protons and Neutrons

Given the previous and somewhat technical discussion about QCD, lets look at what we know exists in nature. We know that the lightest spin

$\frac{1}{2}$ strongly interacting particles that can be made in the laboratory are the proton and neutron. The proton has mass $M_p = 938.27231$ MeV, and $M_n = 939.56563$ MeV There are other properties that have been measured, and can be found in the particle data group compilation[3]. There you can find, for the proton

$$
\begin{aligned}
M_p &= 938.27231 \pm 0.00023 \text{ MeV} \qquad \text{Mass} \\
\mu_p &= 2.79284739 \pm 0.00000006 \mu_N \qquad \text{Magnetic Moment} \\
d_p &= (-4 \pm 6) \times 10^{-4} \text{ e cm} \qquad \text{Electric Dipole Moment} \\
\overline{\alpha}_p &= (12.1 \pm 0.9) \times 10^{-4} \text{ fm} \qquad \text{Electric Polarizability} \\
\overline{\beta}_p &= (2.1 \pm 0.9) \times 10^{-4} \text{ fm} \qquad \text{Magnetic Polarizability} \\
& \tau_p > 1.6 \times 10^{25} \text{ yrs} \qquad \text{Mean Lifetime} \\
r_p &= 0.862 \pm 0.012 \text{ fm} \qquad \text{Charge Radius} \quad ,
\end{aligned} \tag{9}
$$

and for the neutron

$$
\begin{aligned}
M_n &= 939.56563 \pm 0.00028 \text{ MeV} \qquad \text{Mass} \\
M_n - M_p &= 1.293318 \pm 0.000009 \text{ MeV} \\
\mu_n &= -1.9130428 \pm 0.0000005 \mu_N \qquad \text{Magnetic Moment} \\
& d_p < 0.97 \times 10^{-25} \text{ e cm} \qquad \text{Electric Dipole Moment} \\
\overline{\alpha}_p &= (9.8 \pm 2.1) \times 10^{-4} \text{ fm} \qquad \text{Electric Polarizability} \\
\tau_n &= 886.7 \pm 1.9 \text{ s} \qquad \text{Mean Lifetime} \\
r_n^2 &= -0.119 \pm 0.0016 \text{ fm} \qquad \text{Charge Radius} \quad .
\end{aligned} \tag{10}
$$

Some comments:

1. The masses of the neutron and proton are very similar, differing at the 10^{-3} level. Now, perhaps most fortunately for us, the neutron is slightly heavier than the proton, so that $n \to pe\overline{\nu}$ in free space, and not $p \to n\overline{e}\nu$ as would happen if $M_n < M_p$. This scenario would mean that there is no stable hydrogen, and hence a very different universe than the one we live in!

2. So far an electric dipole for neither the proton d_p nor the neutron d_n has been measured. Such a quantity violates time-reversal invariance, a symmetry that we expect to satisfied in nature to a very high precision. However, the present limit on d_n is close to where some extensions to

the standard model of electroweak interactions would may produce a value for d_n. The standard model is estimated to give a d_n a few orders of magnitude lower.

3. It is clear from the fact that both the neutron and proton have non-zero charge radii (for a review see [4]) and also non-zero electric and magnetic polarizabilities, $\overline{\alpha}_p$, $\overline{\alpha}_n$ that the neutron and proton are not point like objects, and have an extended size not so much smaller than 1 fm. A plot of the proton charge form factor can be seen in fig. (2). If the proton wa s apoint object then we would expect the curve to behave like Q^4, which clearly it doesn't!

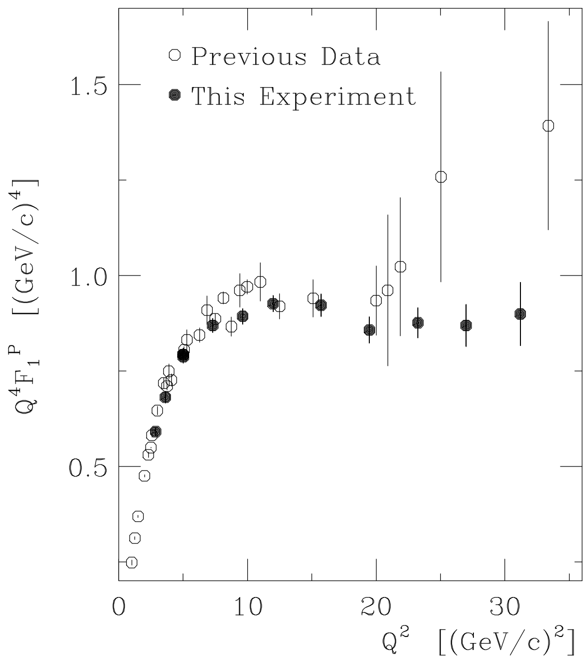

Figure 2: *The proton charge form factor $F_1(Q^2)$ multiplied by Q^4. Data is from the E136 experiment at SLAC (reproduced without permission).*

4.1 The Low-Energy Theory of QCD

The title of this section would lead one to believe that we know the low-energy theory of QCD, when in reality we do not. We know that the hadrons that are cataloged in the particle data book[3] are the asymptotic states of the theory, either mesons or baryons, but in addition all nuclei are also the asymptotic states of QCD. If we could solve QCD we would know the two-body, three-body and n-body interactions, and with these interactions proceed to compute the ground state energy, spin and quadrupole moment of all nuclei. We are not even at the stage where we can use lattice gauge theory to make any predictions at all for systems involving more than one nucleon.

All this sounds very daunting, yet there are still more lectures in this course. For many years, even before the discovery of QCD, nuclear physics was able to make predictions, and allow understanding of phenomena at a relatively good level of precision. If one is interested in observable of a system where the typical, or maximum momentum of the system is much less than 1 GeV, which corresponds to about 1/5 fm, the wavelength of all interacting particles is so large that they cannot resolve the composite structure of each other. What this means is that at low-energies, we need only know "global" properties of the protons and neutrons, like their mass, magnetic moments, charge radii, etc, and this will be sufficient to do a reasonable job in computing many observables to a given precision. We do not need to know *anything* about their structure in terms of quarks and gluons, if we are willing to fit some parameters, as opposed to predict everything. However, this approach is still predictive. Clearly, it will not be sufficient to compute quantities to arbitrary precision, but as we will see, one can do pretty well.

Therefore, we are going to study nucleons and their interactions with themselves and other hadrons. This can be done with no knowledge of QCD. However, we must keep in mind that our low-energy theory (picture) will fail for momentum transfers $\sim$ 1 GeV or so.

4.1.1 Pions

One of the implications of our understanding of QCD and the spontaneous breaking of chiral symmetry, is that there should be a nearly massless strongly interacting spin-zero particle, a pseudo-Goldstone particle. When we say nearly massless, we mean that the mass is much less than typical strong interaction length scales Λ_χ. In the limit that the up and down quark masses were to vanish then this particle would have exactly zero mass. The pion (π) has been identified with this particle and it will play a significant role in our discussions of nuclear physics. There are three different charge states, π^+,

π^0 and the π^-. The properties of the π^+ and π^- are identical by the CPT theorem (Field Theory). In [3] you will find

$$\begin{aligned}
M_{\pi^\pm} &= 139.5679 \pm 0.0007 \text{ MeV} \\
M_{\pi^0} &= 134.9743 \pm 0.0008 \text{ MeV} \qquad \text{Mass} \\
\tau_{\pi^\pm} &= (2.6030 \pm 0.0024) \times 10^{-8} \text{ s} \\
\tau_{\pi^0} &= (8.4 \pm 0.06) \times 10^{-17} \text{ s} \qquad \text{Mean Lifetime} \\
\overline{\alpha}_{\pi^\pi} &= (6.8 \pm 1.4) \times 10^{-4} \text{ fm} \qquad \text{Electric Polarizability} \\
\overline{\beta}_{\pi^\pm} &= -(5.4 \pm 4.5) \times 10^{-4} \text{ fm} \qquad \text{Magnetic Polarizability} \\
r_{\pi^\pm} &= 0.66 \pm 0.03 \text{ fm} \qquad \text{Charge Radius} \quad . \qquad (11)
\end{aligned}$$

It is clear that the π's are also very interesting objects. In fact a large part of nuclear and hadronic physics has been spent better understanding exactly what a pion is! We will not really get into this area during this course, but may step toward that during the next quarter.

5 Isospin

At the quark level we have motivated the existence of a symmetry called isospin. The world is very nearly invariant at the level of strong interactions under the continuous (unitary) transformation that mixes the up and down type quarks. Clearly, when we consider electromagnetic and weak interactions, in which the up and down quarks couple differently, this is not the case, but lets forget this at present.

Turning to the nucleons, and forgetting about weak and electromagnetic interactions, with identical spins and nearly identical masses it is a very good approximation to say that the world would be invariant under continuous, unitary transformations that mix the proton and neutron. Another way of saying this is that the strong Hamiltonian (Lagrangian) is invariant under isospin transformations.

We group the proton and the neutron together into the fundamental representation of the SU(2), and define the isospin $\frac{1}{2}$ nucleon field, N,

$$N_a = \begin{pmatrix} p \\ n \end{pmatrix} \quad , \qquad (12)$$

i.e. $N_1 = p$ and $N_2 = n$. Forgetting about spin, the quantities p and n are annihilation operators, we will return to this shortly. Under isospin transformations

$$N \rightarrow VN \;=\; e^{i\alpha^a T^a}\begin{pmatrix} p \\ n \end{pmatrix} \;=\; \begin{pmatrix} p' \\ n' \end{pmatrix} \quad , \tag{13}$$

where α^a are three real numbers and the T^a are the (hermitian) generators of SU(2), related to the pauli matrices by

$$\begin{aligned} T^a \;&=\; \frac{1}{2}\tau^a \quad , \quad I_2 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \tau^1 \;&=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \tau^2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \tau^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{14}$$

where I_2 is the identity matrix. We have the following nice operations of the matrices

$$\begin{aligned} \tau^3 \begin{pmatrix} p \\ n \end{pmatrix} \;&=\; \begin{pmatrix} +p \\ -n \end{pmatrix} \\ \mathcal{Q}\begin{pmatrix} p \\ n \end{pmatrix} \;&=\; \frac{1}{2}\left(I_2 + \tau^3\right)\begin{pmatrix} p \\ n \end{pmatrix} \;=\; \begin{pmatrix} +p \\ 0 \end{pmatrix} \\ \tau_+ \begin{pmatrix} p \\ n \end{pmatrix} \;&=\; \frac{1}{2}\left(\tau^1 + i\tau^2\right)\begin{pmatrix} p \\ n \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix} \;=\; \begin{pmatrix} n \\ 0 \end{pmatrix} \\ \tau_- \begin{pmatrix} p \\ n \end{pmatrix} \;&=\; \frac{1}{2}\left(\tau^1 - i\tau^2\right)\begin{pmatrix} p \\ n \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ p \end{pmatrix} \quad , \end{aligned} \tag{15}$$

where $\mathcal{Q}$ is the electromagnetic charge matrix, that will be useful later, an $\tau_{+,-}$ are isospin raising and lowering matrices. I have written the above relations in terms of field operators, and in non-relativistic quantum mechanical notation, we define the proton state $|p\rangle$ and the neutron state $|n\rangle$ as

$$\begin{aligned} |p\rangle \;&=\; |I = \frac{1}{2}, I_z = +\frac{1}{2}\rangle \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |n\rangle \;&=\; |I = \frac{1}{2}, I_z = -\frac{1}{2}\rangle \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \end{aligned} \tag{16}$$

where the two-component vector is in isospin space. There are a couple of relations that helps in the manipulation of these two-component objects (both in spin and isospin space)

$$\begin{aligned} \tau^a\tau^b \;&=\; \delta^{ab} I_2 \;+\; i\varepsilon^{abc}\tau^c \\ \left(\tau^a\right)^i_j \;\left(\tau^a\right)^k_l \;&=\; 2\delta^i_l\delta^k_j - \delta^i_j\delta^k_l \quad . \end{aligned} \tag{17}$$

So for instance, if the two-component objects A, B, C, D are comprised of commuting (bosonic) objects, then

$$(A\tau^a B)(C\tau^a d) \;=\; 2(CB)(AD) - (CD)(AB) \quad . \tag{18}$$

Further, for the $I = \frac{1}{2}$ system

$$\tau^2 \;=\; \tau_1^2 + \tau_2^2 + \tau_3^2 \;=\; \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad , \tag{19}$$

which means that the eigenvalues of the $I = \frac{1}{2}$ systems are $+3$.

Let us now consider the π's in the context of isospin. There are three different charge states π^+, π^0 and π^-. They fit simply into a $I = 1$ multiplet π^a, corresponding to $I_Z = +1, 0, -1$ respectively, with

$$\begin{aligned} \pi^+ &= \frac{1}{\sqrt{2}}\left(\pi^1 - i\pi^2\right) \\ \pi^- &= \frac{1}{\sqrt{2}}\left(\pi^1 + i\pi^2\right) \\ \pi^0 &= \pi^3 \quad , \end{aligned} \tag{20}$$

where again these are understood to be field operators. The situation is a little different for these operators, compared to those associated with the nucleon field. The π operators can annihilate or create a *pi*. We will not use this fact much, but keep it in mind.

Fo our purposes, we will only be considering the pion in the context of nucleons, i.e. for nuclear physics purposes, and so it is simpler for us to put the pions into the adjoint representation of SU(2), which is accomplished by contracting the index on the π^a field with the index of the pauli matrices, which defines the Π^a_b field with one up and one down index,

$$\Pi \;=\; \frac{1}{\sqrt{2}}\,\pi^\alpha\tau^\alpha \;=\; \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix} \quad . \tag{21}$$

In terms of states

$$\begin{aligned} |\pi^+\rangle &= |I = 1, I_z = +1\rangle \\ |\pi^0\rangle &= |I = 1, I_z = +0\rangle \\ |\pi^-\rangle &= |I = 1, I_z = -1\rangle \quad . \end{aligned} \tag{22}$$

5.1 Methodologies

There are two equivalent ways that one can think about using isospin. One is the one that is conventionally used in nuclear physics, where one thinks in terms of states, and combines states with tabulated Clebsch-Gordan coefficients (CG's) (which can be found in [3] and other texts). The other approach is to think in terms of the annihilation and creation operators, and generate the CG's via the Wigner-Eckart theorem. Let us examine both methods with a couple of simple examples.

Nice discussions of group theory can be found in references [5] and [6].

5.1.1 $NN\pi$

Let us begin by considering the interaction strengths between nucleons and pions. We would like to know, on the basis of isospin symmetry alone (neglecting isospin breaking) the relative size of the interaction between, say, $pp\pi^0$ and $np\pi^-$, when all other quantum numbers are the same.

Using states we would go about this by asking about the overlap of states of isospin, we start by combining the $N\pi$ states into states to total isospin,

$$\begin{aligned}
|p\pi^0\rangle &= |p\rangle|\pi^0\rangle = |\frac{1}{2},\frac{1}{2}\rangle|1,0\rangle \\
&= \langle\frac{1}{2},\frac{1}{2},1,0|\frac{3}{2},\frac{1}{2}\rangle\ |\frac{3}{2},\frac{1}{2}\rangle\ +\ \langle\frac{1}{2},\frac{1}{2},1,0|\frac{1}{2},\frac{1}{2}\rangle\ |\frac{1}{2},\frac{1}{2}\rangle \\
|p\pi^-\rangle &= |p\rangle|\pi^-\rangle = |\frac{1}{2},\frac{1}{2}\rangle|1,-1\rangle \\
&= \langle\frac{1}{2},\frac{1}{2},1,-1|\frac{3}{2},-\frac{1}{2}\rangle\ |\frac{3}{2},-\frac{1}{2}\rangle\ +\ \langle\frac{1}{2},\frac{1}{2},1,-1|\frac{1}{2},-\frac{1}{2}\rangle\ |\frac{1}{2},-\frac{1}{2}\rangle
\end{aligned} \tag{23}$$

and therefore, the isospin factors of the "matrix elements" $\langle p|p\pi^0\rangle$ and $\langle n|p\pi^-\rangle$ are

$$\begin{aligned}
\langle p|p\pi^0\rangle &= \langle\frac{1}{2},\frac{1}{2},1,0|\frac{1}{2},\frac{1}{2}\rangle = -\frac{1}{\sqrt{3}} \\
\langle n|p\pi^-\rangle &= \langle\frac{1}{2},\frac{1}{2},1,-1|\frac{1}{2},-\frac{1}{2}\rangle = -\sqrt{\frac{2}{3}}\quad ,
\end{aligned} \tag{24}$$

from which we see a relative factor of $\sqrt{2}$. We have used the fact that the strong interaction is a singlet under isospin transformations, and therefore does not modify the isospin structure of the states.

We can arrive at the same result by considering the tensors describing the N and π fields. In the limit we are considering the world is invariant under arbitrary isospin rotations, and therefore, the hamiltonian describing the world is a singlet in iso-space, i.e. no spare isospin indices are left floating around. The most general form for the effective interaction hamiltonian between nucleons and one pion is, considering only isospin structure and using eqs. (21) and (12),

$$\begin{aligned} \mathcal{H} &= \alpha\ N^{a,\dagger}\Pi_a^b N_b \\ &= \alpha\ \left[\frac{1}{\sqrt{2}}p^\dagger p\pi^0\ +\ p^\dagger n\pi^+\ +\ n^\dagger p\pi^0\ -\ \frac{1}{\sqrt{2}}n^\dagger n\pi^0\right] \quad , \end{aligned} \tag{25}$$

where α is a reduced matrix element, independent of the isospin projections as dictated by the Wigner-Eckart theorem. Therefore, the matrix element of this effective hamiltonian between the desired states is

$$\begin{aligned} \langle p|\mathcal{H}|p\pi^0\rangle &= \frac{1}{\sqrt{2}}\alpha \\ \langle n|\mathcal{H}|p\pi^-\rangle &= \alpha \quad , \end{aligned} \tag{26}$$

which gives relative factor of $\sqrt{2}$ and reproduces the result of the first method in eq. (24).

The method of tensors is much more versatile than using tabulated CG's, as it easily extends to other group structures like SU(3) etc. It is also clearly connected to the Wigner-Eckart theorem.

5.1.2 Including Δ's and $\Delta \to N\pi$

As we know the nucleon is not a fundamental object, it will come as no surprise to learn that it has excited states. One lowest lying excitation that couples incredibly strongly to $N\pi$ is the Δ, which is shorthand for Δ^{++}, Δ^+, Δ^0 and Δ^-, an isospin quartet of particles. They have masses $\sim$ 1230 MeV and widths $\Gamma \sim 110$ MeV. When we neglect electromagnetic and weak interactions these hadrons are almost identical, and can be assigned to an irreducible representation of SU(2).

To figure out how the Δ's are assigned to the totally symmetric tensor T_{abc}, with $a, b, c = 1, 2$ (it is easy to show that this has four independent entries, further if it had a component that was antisymmetry under interchange of two indices, then it could be contracted with ϵ^{ab}), we look at the free hamiltonian for non-relativistic particles, which for nucleons is

$$\mathcal{H}_{\rm free} \ = \ N^{a,\dagger}\left[-\frac{\nabla^2}{2M_N}\right] N_a \quad , \tag{27}$$

and for the T_{abc} field is

$$\mathcal{H}_{\rm free} \ = \ T^{abc,\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right] T_{abc} \quad . \tag{28}$$

Explicitly multiplying out eq. (28), and using the symmetry of T_{abc} under interchange of abc we find

$$\begin{aligned}\mathcal{H}_{\rm free} \ &= \ T^{111,\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right] T_{111} \ + \ 3\ T^{112,\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right] T_{112} \\ &+ 3\ T^{122,\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right] T_{122} \ + \ T^{222,\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right] T_{222} \\ &= \Delta^{++\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right]\Delta^{++} \ + \ \Delta^{+\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right]\Delta^{+} \\ &+ \Delta^{0\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right]\Delta^{0} \ + \ \Delta^{-\dagger}\left[-\frac{\nabla^2}{2M_\Delta}\right]\Delta^{-} \quad ,\end{aligned} \tag{29}$$

which tells us that

$$T^{111} \ = \Delta^{++} \ , \ T^{112} \ = \ \frac{1}{\sqrt{3}}\Delta^{+} \ , \ T^{122} \ = \ \frac{1}{\sqrt{3}}\Delta^{0} \ , \ T^{222} \ = \ \Delta^{-} \tag{30}$$

The relation between strong decay rates $\Delta \to N\pi$ can be found be the two methods we have used previously. Let us determine the ratio of decay rates for $\Delta^{++} \to p\pi^+$ and $\Delta^+ \to n\pi^+$. Firstly, using states

$$\begin{aligned}\langle p\pi^+|\Delta^{++}\rangle \ &= \ \left[\langle\frac{1}{2},\frac{1}{2}|\langle 1,1|\right] |\frac{3}{2},\frac{3}{2}\rangle \\ \langle\frac{1}{2},\frac{1}{2},1,1|\frac{3}{2},\frac{3}{2}\rangle\langle\frac{3}{2},\frac{3}{2}|\frac{3}{2},\frac{3}{2}\rangle \ &= \ \langle\frac{1}{2},\frac{1}{2},1,1|\frac{3}{2},\frac{3}{2}\rangle \ = \ +1\end{aligned}$$

$$\begin{aligned}\langle n\pi^+|\Delta^{+}\rangle \ &= \ \left[\langle\frac{1}{2},-\frac{1}{2}|\langle 1,1|\right] |\frac{3}{2},\frac{1}{2}\rangle \\ &= \ \left[\langle\frac{1}{2},-\frac{1}{2},1,1|\frac{3}{2},\frac{1}{2}\rangle\langle\frac{3}{2},\frac{1}{2}| \ + \ \langle\frac{1}{2},-\frac{1}{2},1,1|\frac{1}{2},\frac{1}{2}\rangle\langle\frac{1}{2},\frac{1}{2}|\right] \ |\frac{3}{2},\frac{1}{2}\rangle \\ &= \ \langle\frac{1}{2},-\frac{1}{2},1,1|\frac{3}{2},\frac{1}{2}\rangle \ = \ +\frac{1}{\sqrt{3}} \quad ,\end{aligned} \tag{31}$$

which leads to $\Gamma(\Delta^{++} \to p\pi^+) \ = \ 3\Gamma(\Delta^+ \to n\pi^+)$.

Solving the same problem using tensors we have and effective hamiltonian of the form

$$\begin{aligned}\mathcal{H} \ &= \ \beta \epsilon^{cd} N^{a\dagger} \Pi_c^b T_{abd} \\ &= \ \beta \left[p^\dagger \left(\sqrt{\frac{2}{3}} \Delta^+ \pi^0 \ + \ \frac{1}{\sqrt{3}} \Delta^0 \pi^+ \ - \ \Delta^{++}\pi^- \right) \right. \\ &\quad \left. + \ n^\dagger \left(\sqrt{\frac{2}{3}} \Delta^0 \pi^0 \ + \ \frac{1}{\sqrt{3}} \Delta^- \pi^+ \ - \ \frac{1}{\sqrt{3}} \Delta^+ \pi^- \right) \right] \quad , \end{aligned} \tag{32}$$

where ϵ is the totaly antisymmetric tensor, $\epsilon^{12} \ = \ -\epsilon^{21} \ = \ +1$. It follows that

$$\langle p\pi^+ | \mathcal{H} | \Delta^{++} \rangle \ = \ -\beta \ , \ \langle n\pi^+ | \mathcal{H} | \Delta^+ \rangle \ = \ -\frac{1}{\sqrt{3}} \beta \quad , \tag{33}$$

and therefore $\Gamma(\Delta^{++} \to p\pi^+) \ = \ 3\Gamma(\Delta^+ \to n\pi^+)$.

5.1.3 $\pi - \pi$ Systems

Let us consider the isospin structure of the simplest hadronic system composed of two objects, two pions. The spin J and parity π, which are combined into the usual notation J^π, of the pion is $J^\pi \ = \ 0^-$. It is spinless, and has intrinsically odd-parity. In a system comprised of two pions, the intrinsic parity is $(-1)(-1) = +1$, intrinsically even. Therefore the parity of a two-pion system is given entirely by the parity of the relative orbital angular momentum state they are in $(-1)^L$.

Looking at the isospin structure of a two pion system. Each pion has $I = 1$, and therefore the two pion system can have total isospin $I_{\text{total}} \ = \ 0, 1, 2$. The $I_{\text{total}} \ = \ 0$ and 2 configurations are symmetric under the interchange of the pions, while the $I_{\text{total}} \ = \ 1$ is odd under such an interchange. In terms of tensors, we are able to simply construct the irreducible representations of SU(2) from $\mathbf{3} \otimes \mathbf{3} \ = \ \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$. The objects that transforms as a $\mathbf{5}$ of SU(2)($I \ = \ 2$, a symmetric two-index tensor), as a $\mathbf{3}$ of SU(2)($I \ = \ 1$, a one-index tensor), and as a $\mathbf{1}$ of SU(2)($I \ = \ 0$, a zero-index tensor), are

$$\begin{aligned} T^{ab} &\sim \pi^a \pi^b - \frac{1}{3} \delta^{ab} \pi^\alpha \pi^\alpha \\ T_a &\sim \varepsilon_{abc} \pi^a \pi^b \\ T &\sim \pi^\alpha \pi^\alpha \quad , \end{aligned} \tag{34}$$

where ε_{abc} is the totally antisymmetric tensor. As each of these objects is viewed as a pair of annihilation operators that will act on the appropriate wavefunction and produce the vacuum state, it is easy to write down the isospin wavefunction for two pions in states of pure isospin. In contrast, when acting on a wavefunction of different isospin, the state is destroyed. It is easy to verify that

$$\begin{aligned}
|(\pi\pi); I=0, I_z=0\rangle &= \frac{1}{\sqrt{3}}\left[|\pi^0\pi^0\rangle \ + \ |\pi^+\pi^-\rangle \ + \ |\pi^-\pi^+\rangle\right] \\
|(\pi\pi); I=1, I_z=0\rangle &= \frac{1}{\sqrt{2}}\left[|\pi^+\pi^-\rangle \ - \ |\pi^-\pi^+\rangle\right] \\
|(\pi\pi); I=2, I_z=0\rangle &= \frac{1}{\sqrt{6}}\left[2|\pi^0\pi^0\rangle \ - \ |\pi^+\pi^-\rangle \ - \ |\pi^-\pi^+\rangle\right] \quad . \qquad (35)
\end{aligned}$$

For two pions in a relative S-wave ($L = 0$, parity $+$) or any other even orbital angular momentum state, the requirement of a totally symmetric wavefunction means that they can only be in an $I = 0$ or $I = 2$ state. On the other hand, if the pions are in a relative P-wave ($L = 1$, parity $-$) or any other odd orbital angular momentum state, the requirement of a totally symmetric wavefunction means that they can only be in an $I = 1$ state.

5.1.4 Nucleon Masses

The nucleon masses can be written in terms of irreducible representations of isospin. The effective hamiltonian that describes the nucleon masses is

$$\begin{aligned}
\mathcal{H} &= M_p p^\dagger p \ + \ M_n n^\dagger n \\
&= \frac{1}{2}(M_p + M_n)\left(p^\dagger p + n^\dagger n\right) \ + \ \frac{1}{2}(M_p - M_n)\left(p^\dagger p - n^\dagger n\right) \\
&= \frac{1}{2}(M_p + M_n) N^\dagger N \ + \ \frac{1}{2}(M_p - M_n) N^\dagger \tau^3 N \quad . \qquad (36)
\end{aligned}$$

The first term is called the "isoscalar mass term" (it transforms as a scalar under isospin transformations), and the second is called the "isovector mass term" (it transforms as a vector, τ^a under isospin transformations).

5.1.5 NN Systems

The isospin analysis of the two nucleons sector is similar to the above. Each nucleon is a doublet of SU(2), denoted by **2**, and the isospin structure of the two nucleon system is $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$. Keeping track of only the isospin indices, we see that the $\mathbf{3}$ ($I = 1$, denoted by a symmetric two-index tensor) and the $\mathbf{1}$ ($I = 0$, denoted by a zero-index tensor) are

$$\begin{aligned} T_{ij} &\sim N_i N_j \ + \ N_j N_i \\ T &\sim \varepsilon^{kl} N_k N_l \quad , \end{aligned} \tag{37}$$

and we have not shown the spin structure associated with each object. In order to understand better how things are working, we must include the spin structure. Each element of the nucleon field has an index that denotes the spin, e.g. p^a, where $p^a = p \uparrow$ and $p^2 = p \downarrow$. Therefore, we really need to denote the nucleon field by N_i^a, where the up index is the spin index, and the down index is the isospin index. The anticommutation of the nucleon field (it is a fermion) means that for nucleons in an even relative orbital angular momentum state, the only possible configurations are $I = 1, S = 0$ and $I = 0, S = 1$, denoted by the operators

$$\begin{aligned} T_{ij} &\sim \left[N_i^a N_j^b \ + \ N_j^a N_i^b \right] \varepsilon_{ab} \\ T^{ab} &\sim \left[N_i^a N_j^b \ + \ N_i^b N_j^a \right] \varepsilon^{ij} \quad . \end{aligned} \tag{38}$$

Returning to the isospin structure of the wavefunction of two nucleons, it is easy to see that the $I = 1$ part of two-nucleon wavefunction are

$$\begin{aligned} |(NN), I = 1, I_z = 1\rangle &= |pp\rangle \\ |(NN), I = 1, I_z = 0\rangle &= \frac{1}{\sqrt{2}} \left[|pn\rangle \ + \ |np\rangle \right] \\ |(NN), I = 1, I_z = -1\rangle &= |nn\rangle \quad , \end{aligned} \tag{39}$$

while the $I = 0$ part of a two-nucleon wavefunction is

$$|(NN), I = 0, I_z = 0\rangle = \frac{1}{\sqrt{2}} \left[|pn\rangle \ - \ |np\rangle \right] \quad . \tag{40}$$

The analogy with two-component spin wavefunctions that you will have encountered in nonrelativistic quantum mechanics is exact, for obvious reasons.

As nucleons are fermions, with $J^\pi = \frac{1}{2}^+$, nucleon wavefunctions must be totally antisymmetric under pair-wise interchange. For two nucleons in an even orbital angular momentum state, $L = 0, 2, ...$, the spin-isospin part of the wavefunction must be antisymmetric, as the space part is symmetric, and hence $I = 1, S = 0$ or $I = 0, S = 1$, as mentioned above. The s-wave $I = 1, S = 0$ channel is called the ${}^1S_0$ channel, while the s-wave $I = 0, S = 1$ channel is called the ${}^3S_1$ channel. *Apriori* the physics in these two channels are unrelated, and in fact we will see that they are different, but yet have similarities that are very suggestive of deeper physics. In particular, there is a bound state in the ${}^3S_1(-{}^3D_1)$ channel, while there is not one in the ${}^1S_0$ channel. However, they share a common feature of having very large scattering amplitudes near zero energy.

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