Quantum Mechanics (PHY 517)

Assignment 2 : Bra's, Ket's and Operators

This problem set is due **Thursday October 11**, at the end of the lecture. Feel free to discuss the problems with others in the class, but you must write your own solutions. The "plus" and "minus" below indicate parts are added or removed from the corresponding problem in Sakurai.

1. Sakurai 1 plus : Prove that

$$\begin{aligned} \left[\, \hat{A}\hat{B}, \hat{C}\hat{D} \,\right] &= \hat{A}\{\hat{C},\hat{B}\}\hat{D} - \hat{A}\hat{C}\{\hat{D},\hat{B}\} + \{\hat{C},\hat{A}\}\hat{D}\hat{B} - \hat{C}\{\hat{D},\hat{A}\}\hat{B} \\ \left[\hat{A}, \hat{B}\hat{C}\right] &= \hat{B}\left[\hat{A},\hat{C}\right] + \left[\hat{A},\hat{B}\right]\hat{C} \end{aligned}$$

2. Sakurai 4 minus : Using the rules of bra-ket algebra, prove or evaluate
   (a) $\mathrm{Tr}\left[\, \hat{X}\hat{Y} \,\right] = \mathrm{Tr}\left[\, \hat{Y}\hat{X} \,\right]$
   (b) $\left(\hat{X}\hat{Y}\right)^\dagger = \hat{Y}^\dagger\hat{X}^\dagger$
   (c) What is $e^{if(\hat{A})}$ in terms of the eigenkets of $\hat{A}$, where $\hat{A}$ is a hermitian operator with eigenkets $|a_i\rangle$ and $f(x)$ is a smooth function that can be expanded in a power-series.

3. Sakurai 5 plus :
   (a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Given that the $\{|a_i\rangle\}$ form a complete set of basis kets and that the inner products $\langle a_i|\alpha\rangle$ and $\langle a_i|\beta\rangle$ are known, find the matrix representation of the operator $|\beta\rangle\langle\alpha|$ in the basis of $\{|a_i\rangle\}$.
   (b) What is the matrix representation of $|s_y = -\frac{\hbar}{2}\rangle\langle s_x = -\frac{\hbar}{2}|$ in the basis where $\hat{S}_z$ is diagonal?

4. Sakurai 6 : If the kets $|\alpha\rangle$ and $|\beta\rangle$ are eigenkets of the hermitian operator $\hat{A}$, under what conditions will the sum $|\alpha\rangle + |\beta\rangle$ also be an eigenkets of $\hat{A}$? Explain.

5. Sakurai 7 minus : Consider an N-dimensional ket space spanned by the N eigenkets $\{|a_i\rangle\}$ (where $i = 1, .., N$) of a hermitian operator $\hat{A}$. Assume that the eigenvalues a_i are nondegenerate.

(a) What ket(s) are produced by the action of the operator

$$\hat{\theta} \;=\; \Pi\left(\hat{A} - a_i\right) \;=\; \left(\hat{A} - a_1\right)\left(\hat{A} - a_2\right)...\left(\hat{A} - a_N\right)$$

on ANY arbitrary ket in the ket space.

(b) Setting $\hat{A} = \hat{S}_x$ and acting in the ket space associated with a spin-$\frac{1}{2}$ particle, demonstrate your answer to part (a).

(c) What is the comutator of $\hat{\theta}$ with itself?

6. Let A and B be $n \times n$ matrices.

   (a) Show that for $\det(A) = 0$ no matrix B exists for which $AB = I$ or $BA = I$ where I is the identity matrix.

7. Let $\Delta(\lambda) = \det(A(\lambda))$. Show that

$$\frac{d}{d\lambda}\Delta(\lambda) \;=\; \sum_{k=1}^{n} \Delta_k(\lambda) \quad , \tag{1}$$

where $\Delta_k(\lambda)$ is the determinant formed by replacing the $k-th$ row of $A(\lambda)$ by the derivative of the $k-th$ row.