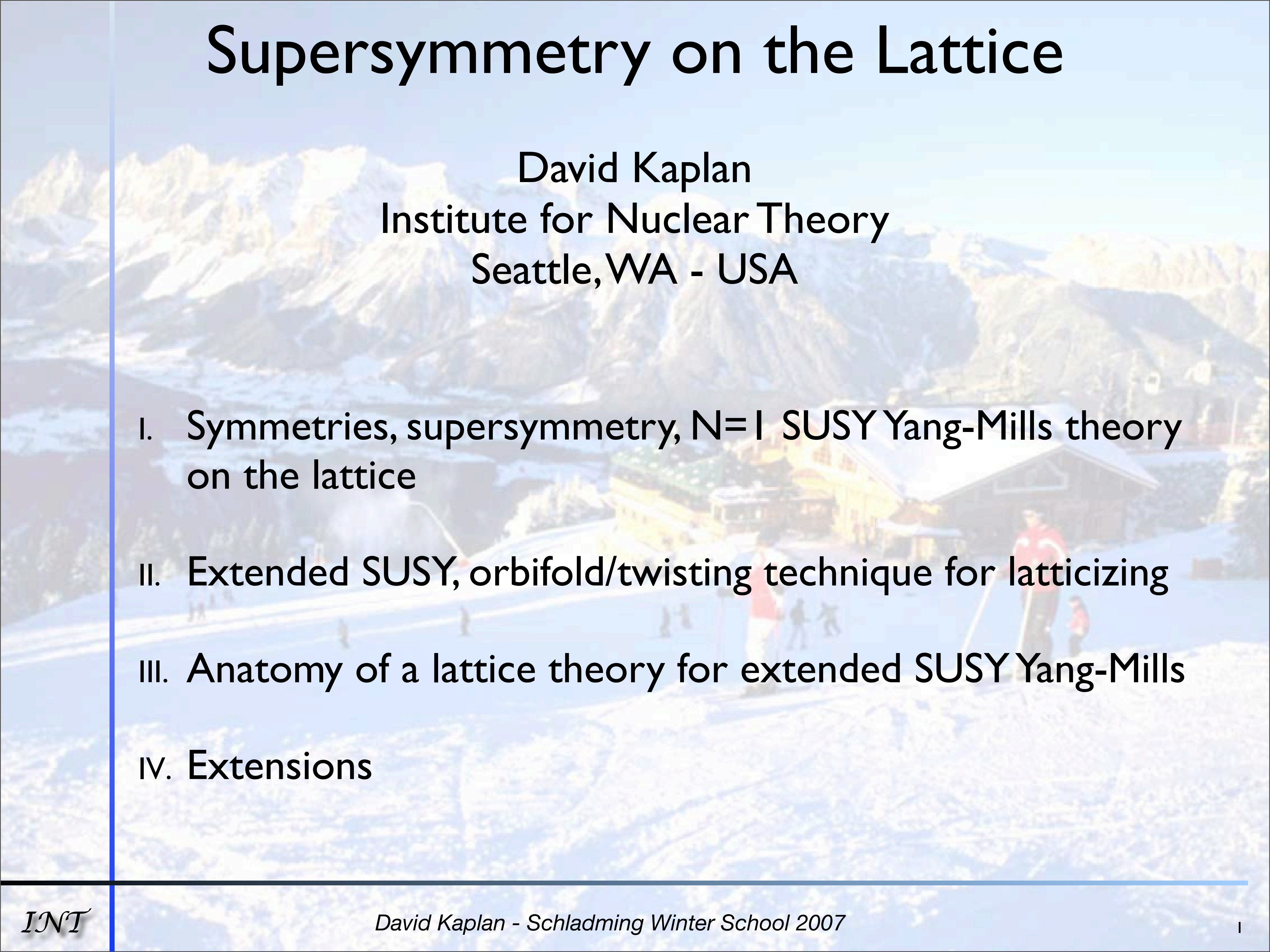

Supersymmetry on the Lattice

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I. Symmetries, supersymmetry, N=1 SUSY Yang-Mills theory on the lattice

II. Extended SUSY, orbifold/twisting technique for latticizing

III. Anatomy of a lattice theory for extended SUSY Yang-Mills

IV. Extensions

Part I:

Relevance & symmetry

- Relevance
- Naturalness
- Symmetry
- Accidental symmetry

I. Relevance and symmetry

Operators are classified by how they scale in the IR:

- Irrelevant: less important in the IR
- Relevant: more important in the IR
- Marginal: scale invariant

At the classical level, operators are marginal if their mass dimension equals the spacetime dimension.

Lower operator dimension = more relevant.

Example: $(\bar{\psi}\psi)^2$ ~ mass dimension = 6 (irrelevant)

$$\sigma_{\nu e\to\nu e} \propto G_F^2 E^2 \longrightarrow 0 \quad \text{as} \quad E \longrightarrow 0$$

Quantum corrections change the dimensions of operators

For a generic weakly coupled theory:

- Only small effects for relevant or irrelevant operators
- A large effect on marginal operators

Example: QCD interaction is marginal (dimensionless coupling constant) at the classical level, but relevant at one-loop (asymptotic freedom).

Conformal field theories:

- Scale invariant
- Marginal operators, whose engineering dimension may be far from the spacetime dimension if the theory is strongly coupled.

Example: N=4 Supersymmetric Yang-Mills theory

Naturalness

Old fashioned view:

- Irrelevant operators are bad (**nonrenormalizable!**)
- Relevant operators are great (superrenormalizable!)

Modern view:

- Irrelevant operators are fine (irrelevant!)
- Relevant operators are baffling

(**particles with relevant interactions should be too heavy to see!**)

Example:

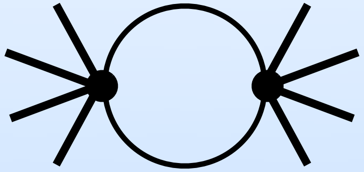
dimensionless coupling

scalar field

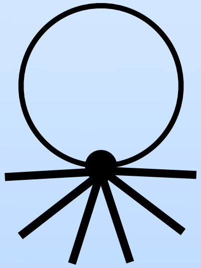
$$\mathcal{L}_6 = \frac{g}{\Lambda^2}\phi^6$$

UV momentum cutoff

If you also have: $\mathcal{L}_8 = \frac{\kappa}{\Lambda^4}\phi^8$, then the two operators renormalize each other at one loop:



$$\delta\kappa \sim \frac{g^2}{16\pi^2}\ln\Lambda'/\Lambda$$



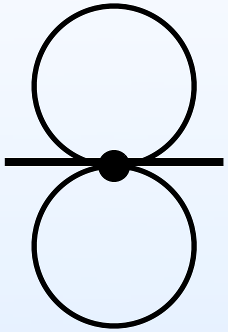
$$\delta g \sim \frac{\kappa}{16\pi^2}$$

Couplings are "natural" if $g \gtrsim \delta g\,,\ \ \kappa \gtrsim \delta\kappa\ldots$

But now consider a mass term (relevant operator):

$$\mathcal{L}_2 = c\Lambda^2\phi^2$$

One quantum correction:



$$\delta c \sim \frac{g}{(4\pi)^4}$$

Note that $m_\phi \lesssim \Lambda$; can't have $m_\phi \lll \Lambda$ unless:

1. All interactions are extremely weak, or
2. Tree level value + radiative corrections miraculously cancel

To have light, interacting particles, relevant operators have to be "unnaturally" small

Symmetry

Relevant operators can sometimes have naturally small coefficients due to symmetries

Example 1:

$$\mathcal{L}_{\text{bare}} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}c\Lambda^2\phi^2 - \epsilon\Lambda\phi^3 - \lambda\phi^4$$

It is natural to have $\epsilon \lll 1$ because the ϕ^3 violates $\phi \to -\phi$ symmetry, implying that ϵ must be multiplicatively renormalized...

...but unnatural to have c << 1

Example 2:

$$\bar{\psi}\, i\, \not{D} \psi - m\bar{\psi}\psi$$

Approximate chiral symmetry!

Dirac fermion: mass term can be naturally small, because it violates chiral symmetry, and is therefore multiplicatively renormalized.

Example 3:

No approximate chiral symmetry!

$$\bar{\psi}\, i\, \not{D} \psi - m\bar{\psi}\psi - ra\bar{\psi}\Delta\psi$$

Wilson fermion: "Irrelevant" Wilson term violates chiral symmetry; fermion mass must be fine-tuned to be << 1/a

Example 4: Supersymmetry

Boson-fermion symmetry relates fermion mass term to scalar mass term

$$m\bar{\psi}\psi \longleftrightarrow m^2\phi^2$$

Protected by chiral symmetry, so that fermion mass can be naturally small.

Supersymmetry requires boson and fermion to be degenerate...so scalar mass can be naturally small too. (Radiative corrections to scalar mass cancel)

Accidental symmetry

Have seen: symmetry controls relevant operators

Converse: allowed relevant operators determine the symmetries in the IR. Symmetries in the IR which are not symmetries in the UV = "accidental symmetries"

Example 1:

Baryon number symmetry is accidental in the Standard Model: lowest dimension B-violating operator allowed by Lorentz x gauge symmetries has 3 quarks + 1 lepton: $\boxed{qqq\ell}$

Dimension 6 & irrelevant → baryon number is a good approximate symmetry in IR, even if not in UV (eg, SU(5) in UV)

Example 2:

Lorentz symmetry emerges as an accidental symmetry in lattice QCD.

Lattice breaks Lorentz symmetry down to 4d cubic crystal group...but no relevant operator consistent with gauge symmetry x crystal symmetry breaks Lorentz symmetry...

So the IR (continuum) limit is Lorentz invariant!

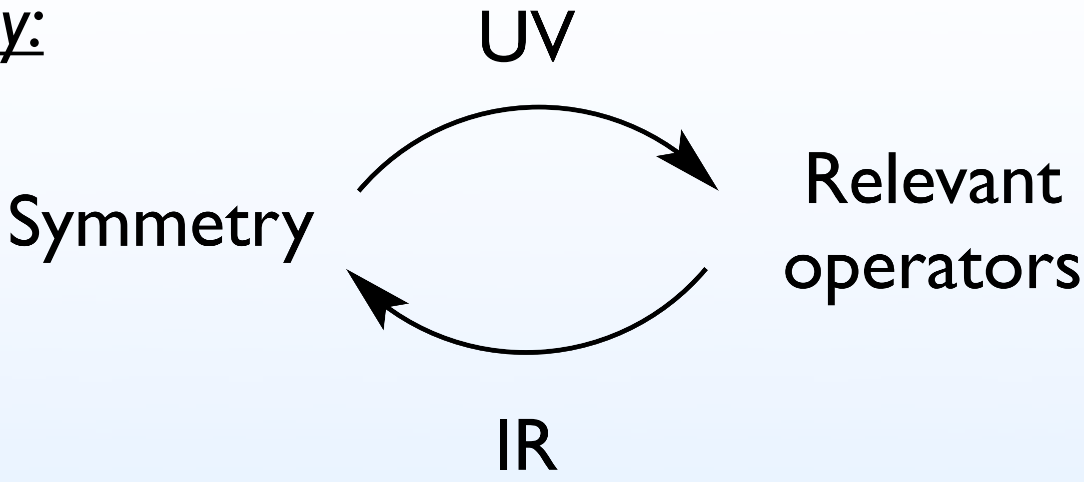
E.g: $A_1 A_2 A_3 A_4$

$A_\mu \equiv \text{gauge field}$

Violates Lorentz symmetry

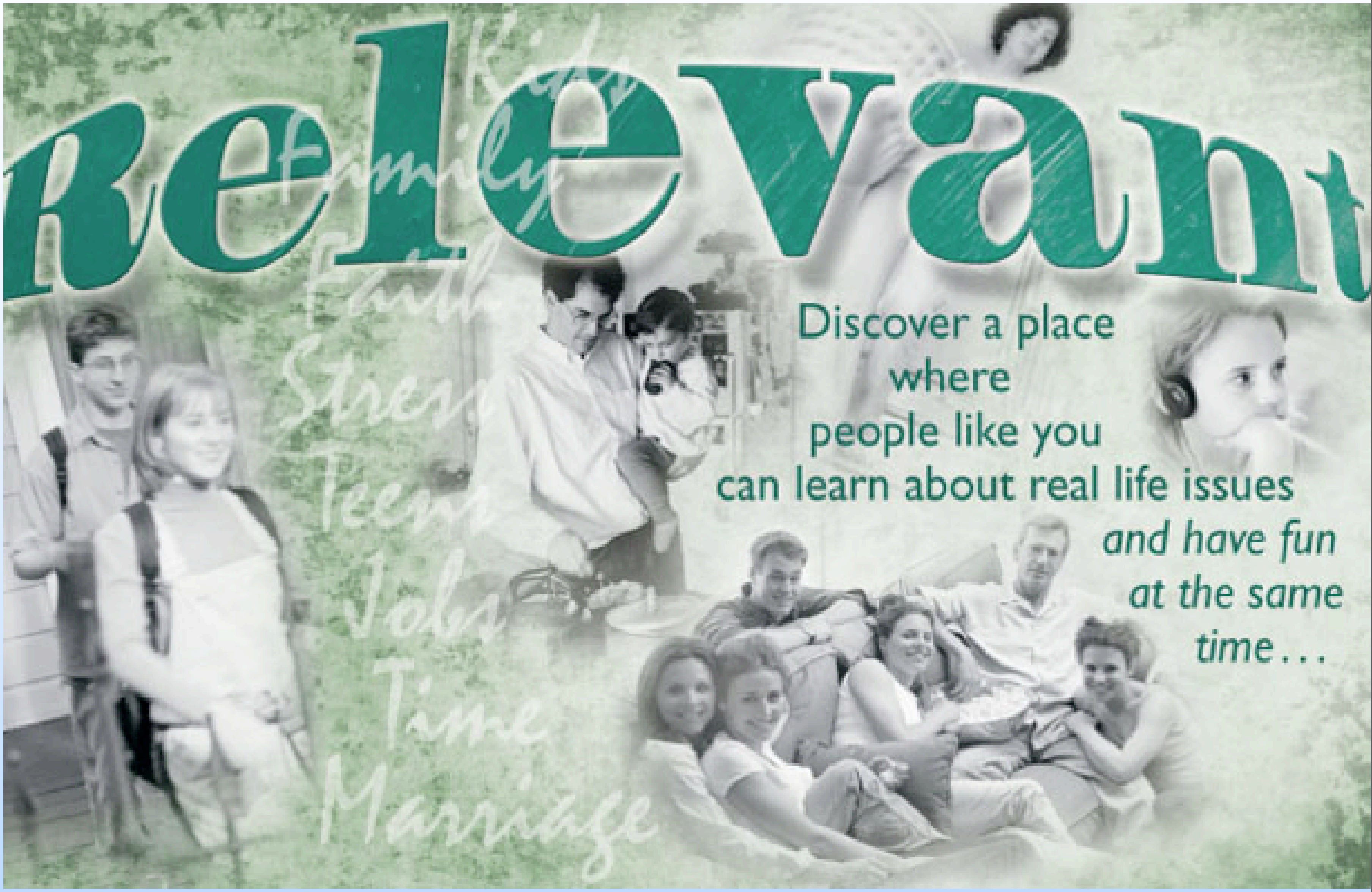
- Consistent with cubic symmetry
- ...but violates gauge symmetry

Summary:



- Irrelevant operators disappear in IR
- Unprotected relevant operators are "un-natural" (eg, Higgs mass, cosmological constant)
- Exact or softly broken symmetries can protect relevant operators and make them "naturally" small or zero.
- Enhanced "accidental symmetries" can emerge in the IR if they can't be broken by allowed relevant & marginal operators.

Relevance & symmetry



Part II.

Supersymmetry

- What & Why
- N=1 SUSY Yang-Mills
- Lattice SUSY
- Accidental SUSY
- Lattice SUSY Yang-Mills

II. Supersymmetry

What & Why

Supersymmetry is a generalization of Poincare symmetry, which relates bosons and fermions

Poincare group generators: $P_\mu, \; \Sigma_{\mu\nu}$

Algebra:

$$[P, P] = 0, \qquad [P, \Sigma] \sim P, \qquad [\Sigma, \Sigma]$$

$P =$ *4-vector* $\qquad \Sigma =$ *a.s. tensor*

Super-Poincare: $Q_\alpha, \; \bar{Q}_{\dot{\alpha}}$

$$\{Q, \; Q\} = 0$$

Grassmann,

$$[Q, \; \Sigma] \sim Q$$

LH Weyl spinor

$$\left\{Q, \; \bar{Q}\right\} \sim P$$

Supersymmetry is interesting because:

- Supersymmetry can protect relevant operators (such as the Higgs mass) from large radiative corrections
- Can study many interesting features of strongly coupled SUSY analytically
  - ❖ chiral symetry breaking
  - ❖ confinement & magnetic monopole condensation
  - ❖ massless composite fermions
- Consequence of superstring theory
- Large-N_c gauge theories related to supergravity & string theory

N=1 SUSY Yang-Mills

In d=4 dimensions, minimal SUSY called "N=1"
One complex Weyl fermion supercharge Q

Supersymmetric Yang-Mills theory (no matter):

"vector supermultiplet": one gauge boson v_m (2 helicities) plus one Weyl fermion gaugino λ_α (2 helicities), both adjoints of the gauge group

$$\mathcal{L} = \bar{\lambda}\, i\bar{\sigma}^m D_m \lambda - \frac{1}{4}\mathrm{v}_{mn}\mathrm{v}^{mn}$$

$$\bar{\sigma}^m = \{1, -\vec{\sigma}\}$$

Assume gauge group SU(N)

- Classical action has a U(1) symmetry acting on the gaugino:

$$\lambda \to e^{i\alpha}\lambda$$

(this symmetry does not commute with the supercharges Q, since v_m does not transform, and so it is called an "R"-symmetry)

- This U(1) R-symmetry is broken to a Z_{2N} symmetry by anomalies: $\alpha = 2\pi n/(2N), \; n = (1, 2, \ldots, 2N)$
- Theory is asymptotically free; gauginos condense, spontaneously breaking the Z_{2N} R symmetry
- Condensate, string tension, domain wall tension can be analytically related

Lattice SUSY

Can we study SUSY on the lattice? Obstacles:

- Supersymmetry will not be preserved on the lattice

SUSY algebra:
$$\left\{Q_\alpha,\ \bar{Q}_{\dot\alpha}\right\} = 2\sigma^m_{\alpha\dot\alpha}\,P_m$$

P = generator of infinitesimal translations...not a symmetry of the lattice

- Gauge bosons, scalars and fermions are treated so *differently* on the lattice:

(i) Gauge bosons on **links**

(ii) scalars on **sites**

(iii) fermions on **sites** (Wilson), or **hypercube** (staggered) or **5th dimension** (DWF)...

Poincare symmetry emerges as accidental symmetry...

Can SUSY emerge as an accidental symmetry in the IR?

Accidental SUSY Yang-Mills

Accidental supersymmetry looks difficult: scalars, fermions, gauge bosons are treated so differently on the lattice.

But, the point of these lectures: **Yes**, SUSY can emerge as an accidental symmetry of the lattice.

Start with a SUSY theory without scalars - N=1 SUSY Yang-Mills in d=4 dimensions in the continuum:

$$\mathcal{L} = \bar{\lambda}\, i\bar{\sigma}^m D_m \lambda - \frac{1}{4} v_{mn} v^{mn}$$

What relevant interactions could be added that would *spoil* the SUSY, consistent with Lorentz & gauge symmetry?

$$\mathcal{L} = \bar{\lambda}\, i\bar{\sigma}^m D_m \lambda - \frac{1}{4} v_{mn} v^{mn}$$

The only relevant operator that can be added to this Lagrangian is a gaugino mass term:

$$\delta\mathcal{L} = m\lambda\lambda + h.c.$$

The gaugino mass breaks:

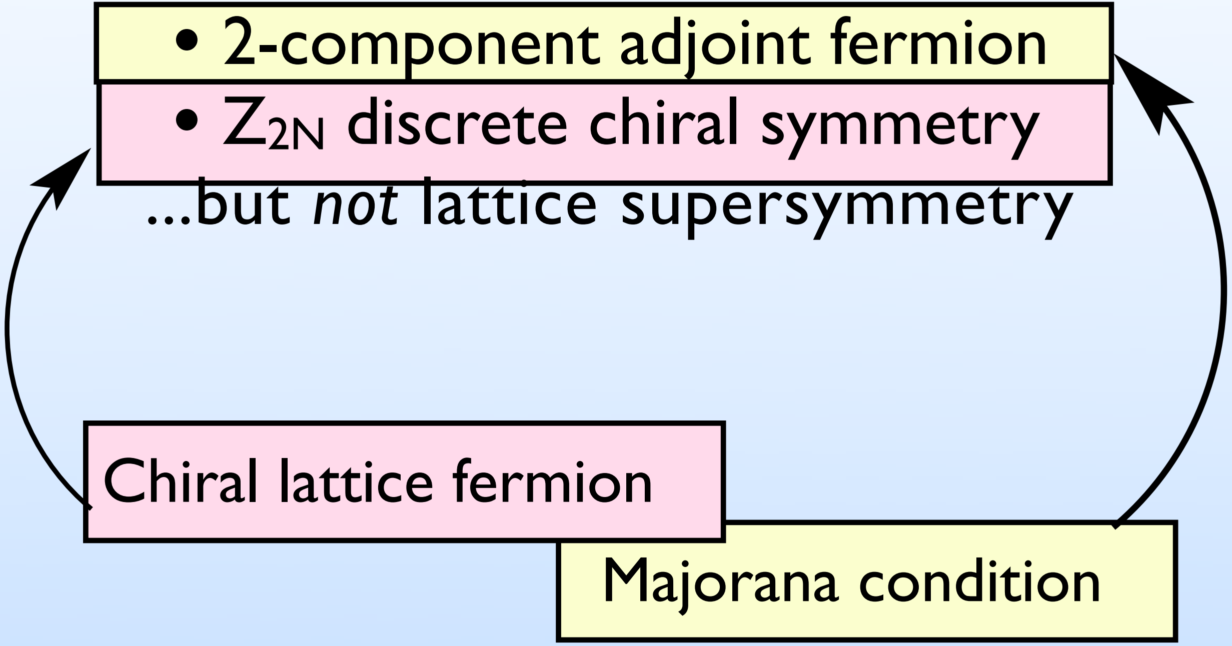
- Supersymmetry
- Z_{2N} chiral symmetry (the R-symmetry)

...so imposing a Z_{2N} chiral symmetry on the theory forbids the gaugino mass, and the IR theory is *accidentally* supersymmetric! (D.K., 1984) ☞ *(my first paper!)*

Lattice SUSY Yang Mills:

Need to construct a lattice theory with:

- SU(N) gauge symmetry



- 2-component adjoint fermion
- Z_{2N} discrete chiral symmetry

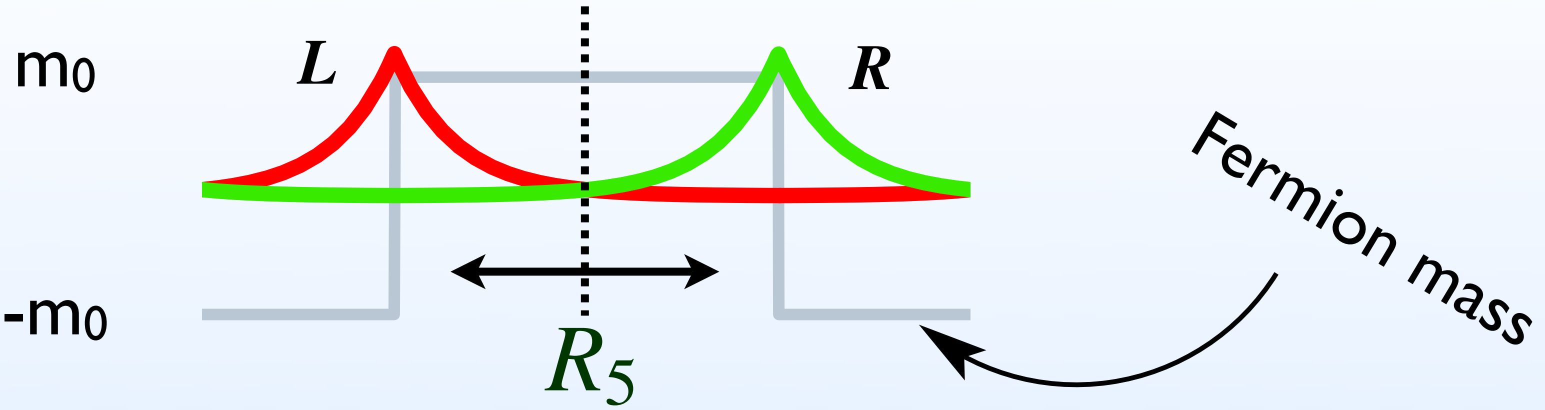
...but *not* lattice supersymmetry

Chiral lattice fermion

Majorana condition

D.K., M. Schmaltz (2000): used domain wall fermions to construct a lattice theory for supersymmetric YM theory

Domain wall fermion



= massless 4d Dirac fermion

$$\Psi = \begin{pmatrix} \alpha \\ \bar{\beta} \end{pmatrix} \qquad \bar{\Psi} = \begin{pmatrix} \bar{\alpha}^T & \beta^T \end{pmatrix}$$

5d fermion

Zero-mode components

"Majorana" constraint: $\Psi = R_5 C \bar{\Psi}^T \longrightarrow \alpha = \beta$

$$\Psi = R_5 C \bar{\Psi}^T$$

only possible because fermion is in a real representation of the gauge group (adjoint)

$$\bar{\Psi}\,\not{D}\Psi \longrightarrow \Psi^T (R_5^T C^T\,\not{D})\Psi$$

$$\det(\not{D}) \longrightarrow \mathrm{Pf}(\underline{R_5^T C^T\,\not{D}})$$

Pfaffian

Antisymmetric matrix

The Pfaffian is an analytic square root of the Dirac operator.
Is there a fermion sign problem?

First, consider the 4d continuum Pfaffian for an adjoint fermion:

$$\mathrm{Pf}[C \not{D}] = \sqrt{\det \not{D}}$$

Look at eigenvalue equation

$$\not{D}\psi = \lambda\psi \;, \quad \not{D} = -\not{D}^{\dagger}$$

λ is imaginary

$$\not{D}\gamma_5\psi = -\gamma_5\psi$$

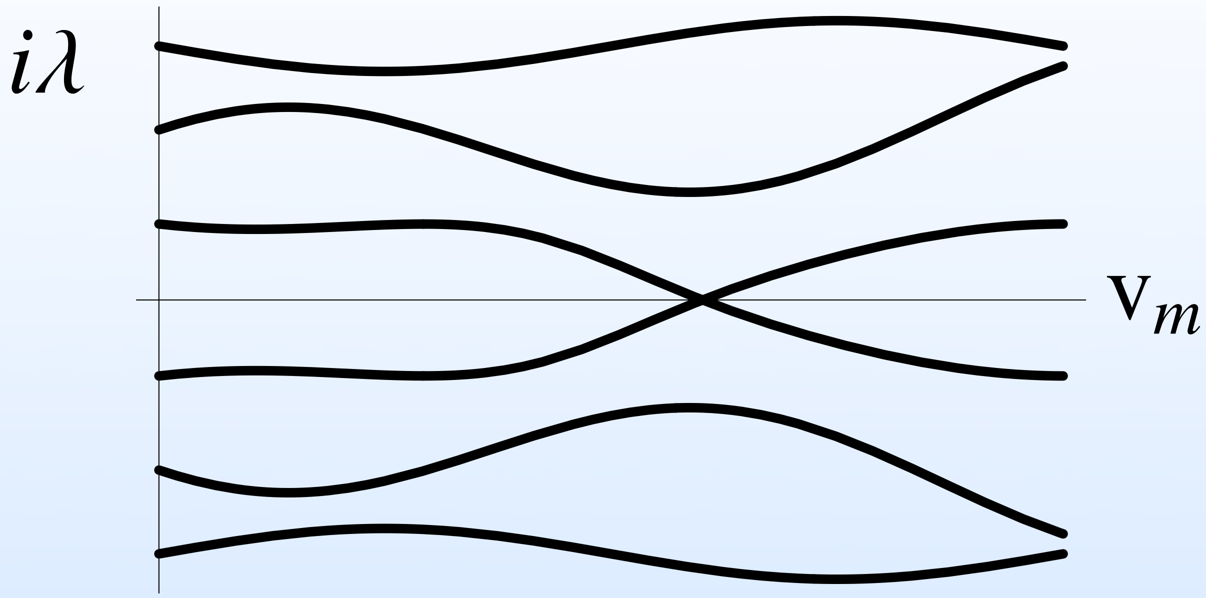
λ comes in ± pairs

$$\not{D}C\psi^* = \lambda C\psi^* \;, \quad \langle\psi|C\psi^*\rangle = 0$$

λ comes in degenerate pairs

(because C is antisymmetric)

Eigenvalues of the Dirac operator for adjoint fermion



Each spectral line has even degeneracy

Therefore can define $\sqrt{\not{D}}$

which is real & positive for all gauge fields

Can similarly show that for 5d domain wall fermions

- $\mathrm{Pf}[R_5 C \not{D}]$ is real, positive in continuum
- Lattice analogue is real, positive at finite a

Neuberger (1997), Kikkukawa

Simulations are hard! (massless dynamical domain wall fermions). Early attempt: Fleming, Kogut, Vranas (2000).

People should return to studying this system!

Epilog:

Is “accidental” supersymmetry necessary? Can’t one just use Wilson fermions and fine-tune away the relevant gaugino mass?

Suffers from fermion sign problem!

This approach has been tried by Montvay & collaborators...not particularly successful. Definitely not a recommended approach for more complicated SUSY theories, with more fine-tuning.

Next: SUSY with scalars and deconstruction

Part III.

Accidental SUSY with scalars

- Recap
- Accidental SUSY requires exact SUSY!
- Why it looks impossible

I. Accidental SUSY with scalars

We have considered N=1 SYM theory:

- SU(N) gauge theory with one Weyl fermion λ in adjoint representation
- Only relevant operator is a fermion mass:

$$m\,\lambda\lambda$$

Violates both SUSY & discrete Z_{2N} chiral symmetry

- Realize Z_{2N} symmetry on the lattice, and SUSY follows "accidentally"

Accidental SUSY requires Exact SUSY

More complicated supersymmetric theories have scalars. A challenge for latticization!

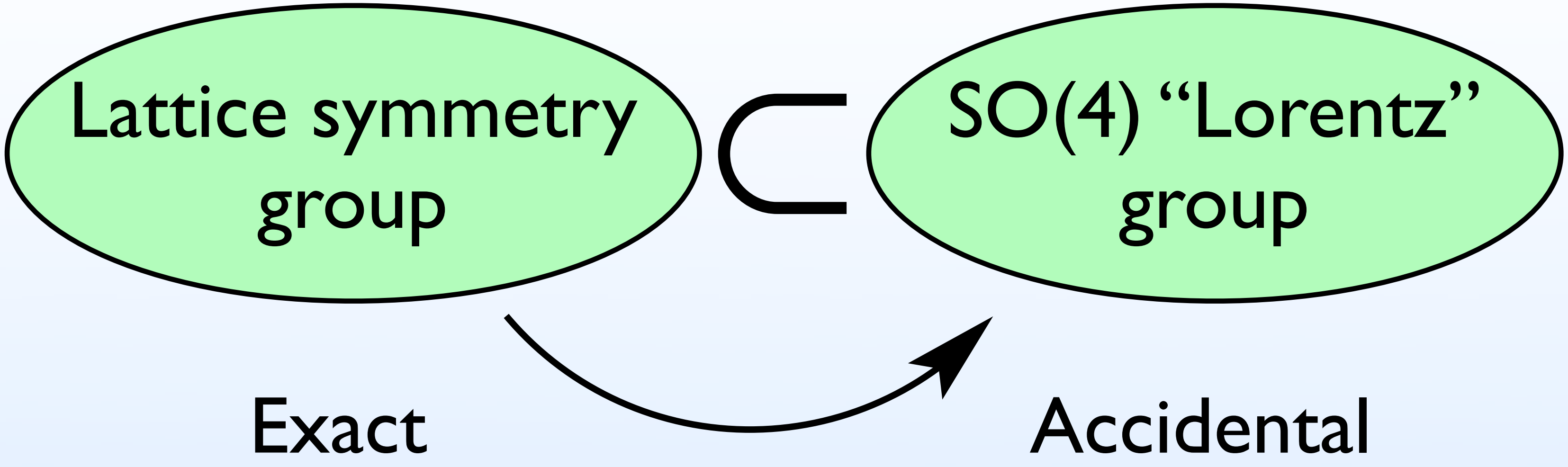
- Scalar mass is a relevant operator that violates SUSY:

$$m^2|\phi|^2$$

...but typically it violates no other symmetry (except a shift symmetry, which only applies to Goldstone bosons)

- Implication: need to implement exact SUSY on the lattice to forbid relevant operator which violates SUSY ?!

Analogy with Lorentz invariance & the lattice:

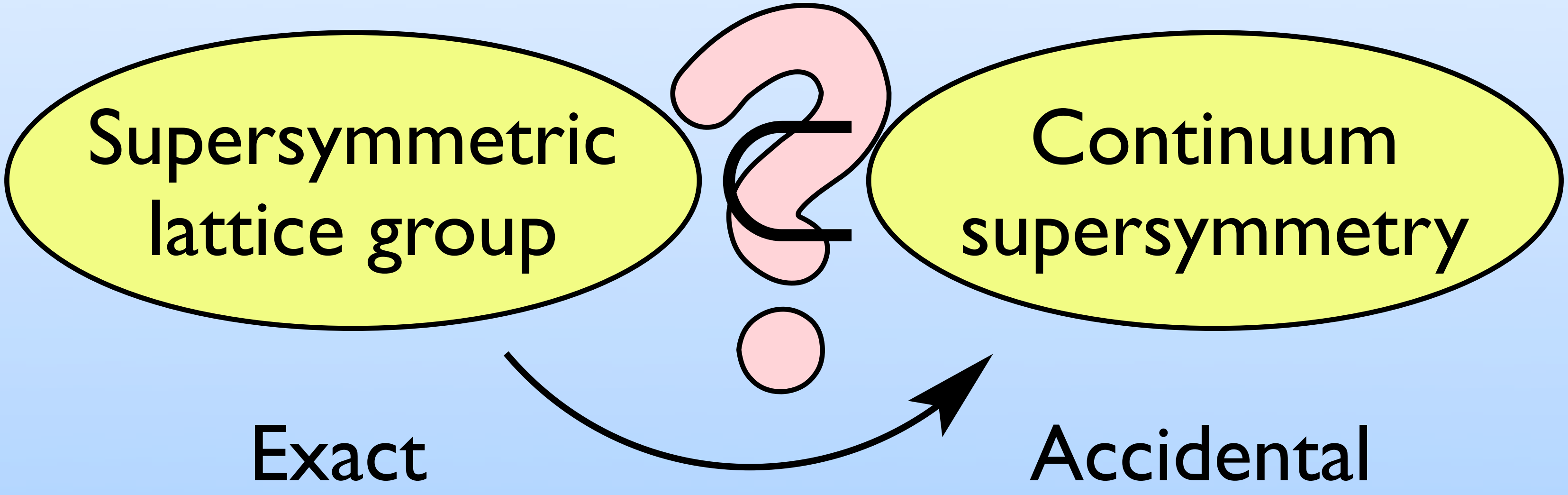


Lattice symmetry group ⊂ SO(4) “Lorentz” group

Exact

Accidental

Lattice subgroup of supersymmetry?



Supersymmetric lattice group ⊂ Continuum supersymmetry

Exact

Accidental

Supersymmetry is not a classical group; rotation "angles" are Grassmann.

"Finite supersymmetry transformations" analogous to "finite translations" or "finite rotations" do not exist. No discrete sub*group* of supersymmetry.

Can consider a sub*algebra* of the full SUSY algebra

$$\left\{Q_\alpha,\, Q_\beta\right\} = 0\ ,\ \ \left\{Q_\alpha,\, \bar{Q}_{\dot\beta}\right\} = 2P_m\sigma^m_{\alpha\dot\alpha}\ ,$$

But which subalgebra?
How to avoid ruining Lorentz symmetry?

Why it looks impossible

- Except for the special case of SUSY without scalars, only exact SUSY can prevent the appearance of SUSY violating relevant operators
- There is no exact SUSY subgroup we can impose
- No guide to how to pick a subalgebra of SUSY
- Exact SUSY on the lattice seems impossible anyway: scalars, fermions, gauge bosons are treated so differently

For example: N=4 SUSY on the lattice:

1 gauge field, 4 Weyl fermions, 6 real scalars

Exact SUSY, if it commutes with the gauge symmetry, implies each of these fields must live at same part of lattice (eg, site, link...)

How can scalars live on links? They will transform nontrivially under rotations by 90 degrees...won't be scalars in the continuum!?

Part IV.

Deconstruction

- The lesson from the 5th dimension
- The AHCG model

A lesson from deconstruction

Arkani-Hamed, Cohen, Georgi: “Deconstruction”

One of their models:

- 4D, *N*=1 supersymmetric gauge theory with gauge group SU(k)^N and spontaneous symmetry breaking at scale $\langle\phi\rangle = 1/\sqrt{2}\,a$
- Continuum 4D, discrete 5th dimension, lattice spacing a
- $N \to \infty,\ a \to 0,\ g \to \infty,\ g^2/a$ fixed: fifth dimension continuous
- → ***5D Lorentz invariant, supersymmetric.***
- → ***5D theory has 8 real supercharges, twice the number of the 4D theory!***

Deconstruction

Lesson from the 5th dimension

The AHCG model

The AHCG model

equal gauge couplings g

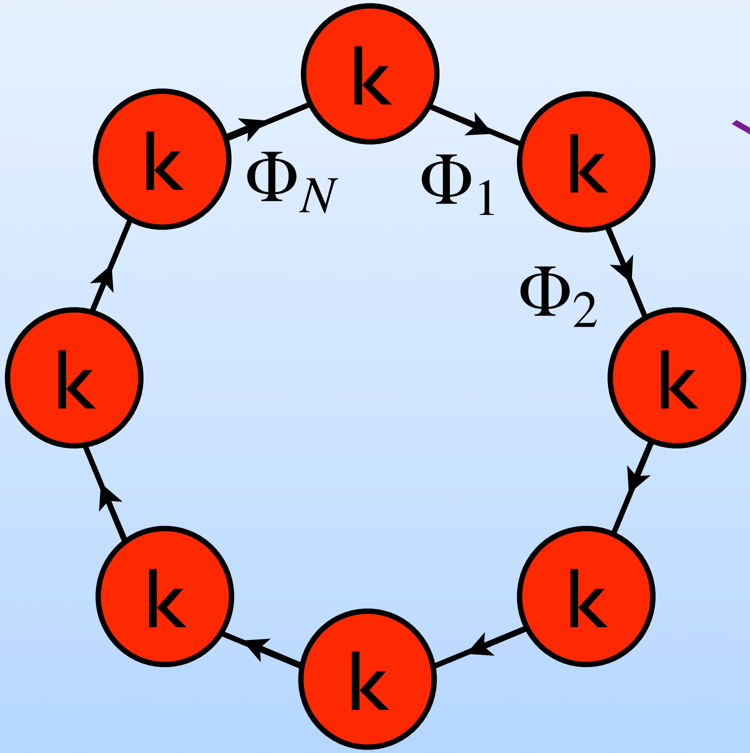
Gauge group: $$G = \underbrace{SU(k) \times \ldots \times SU(k)}_{N}$$

Chiral superfields: $\Phi : \{\phi,\ \psi,\ F\}$ **k x k matrices**

scalar

Weyl fermion

Auxiliary field



SU(k) x SU(k)x...

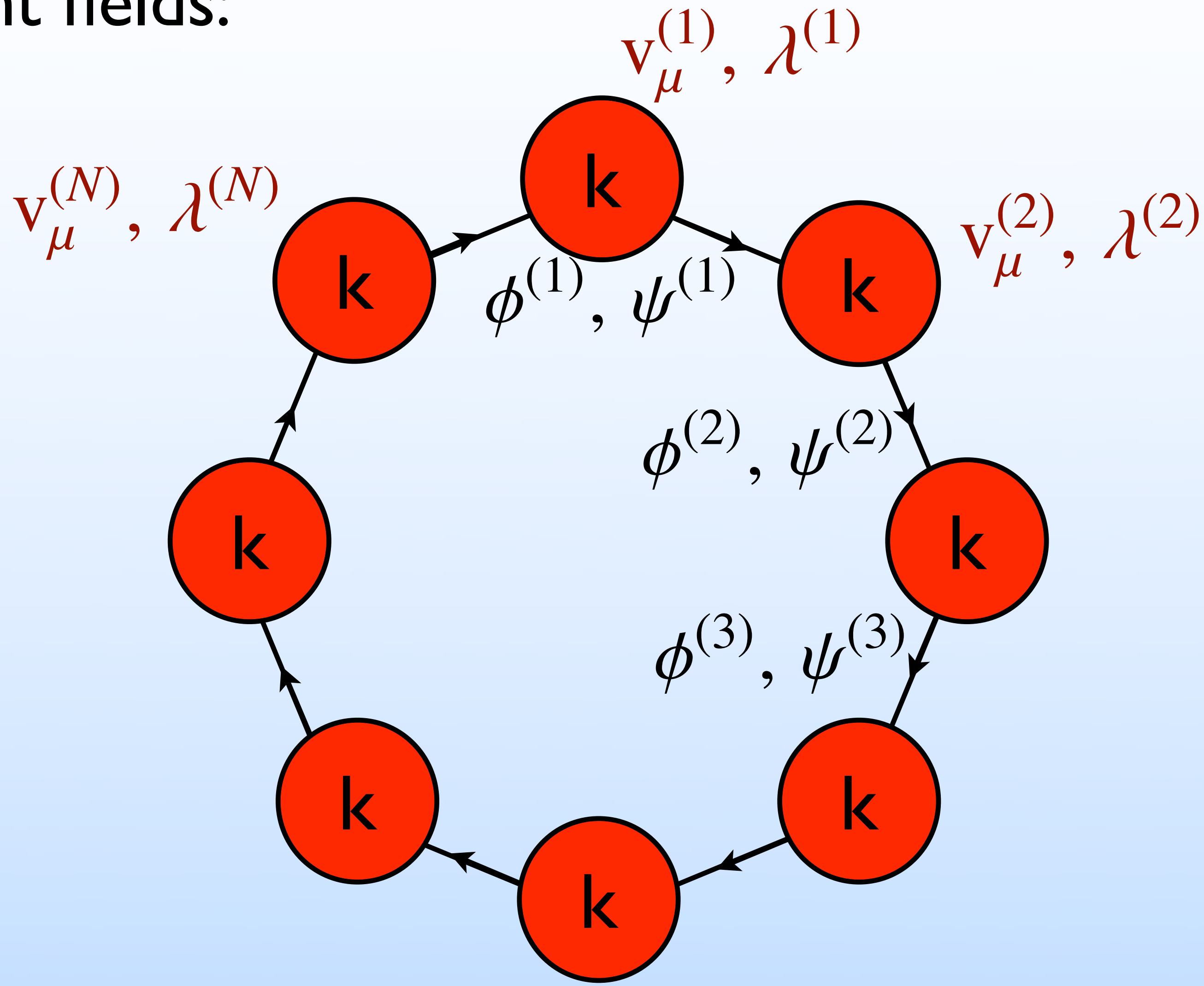
$$\Phi_1 = (k, \bar{k}, 1, 1, \ldots)$$

$$\Phi_2 = (1, k, \bar{k}, 1, \ldots)$$

$$\vdots$$

$$\Phi_N = (\bar{k}, 1, 1, \ldots, k)$$

Component fields:

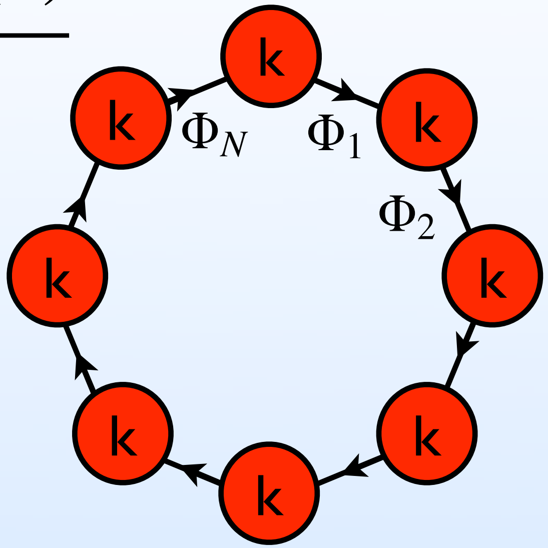


Gauge & gaugino fields at the sites: U(k) adjoints

Scalars & fermions on links: U(k)xU(k) bifundamentals

Model has degenerate vacua. Expand about

$$\phi^{(n)}(x) = \frac{1}{\sqrt{2}a}\mathbf{1}_k + \frac{s^{(n)}(x) + i v_5^{(n)}(x)}{\sqrt{2}}$$



and treat as lattice 5th dimension with lattice spacing $a \to 0$

Examples of terms in action:

1. Scalar kinetic term ($\mu = 1,\ldots,4$)

$$\sum_n \left|D_\mu \phi^{(n)}\right|^2 = \sum_n \left|\partial_\mu \phi^{(n)} + i v_\mu^{(n)} \phi^{(n)} - i \phi^{(n)} v_\mu^{(n+1)}\right|^2$$

x5 hopping term for gauge field

Deconstruction

Lesson from the 5th dimension

The AHCG model

$$\frac{1}{g^2}\sum_n \left|D_\mu\phi^{(n)}\right|^2 = \frac{1}{g^2}\sum_n \left|\partial_\mu\phi^{(n)} + i\mathrm{v}_\mu^{(n)}\phi^{(n)} - i\phi^{(n)}\mathrm{v}_\mu^{(n+1)}\right|^2$$

Substitute: $\phi^{(n)}(x) = \frac{1}{\sqrt{2}\,a}\mathbf{1}_k + \frac{s^{(n)}(x) + i\mathrm{v}_5^{(n)}(x)}{\sqrt{2}}$

$$= \frac{1}{2g^2}\sum_n \mathrm{Tr}\left|\boxed{\partial_\mu s^{(n)} + i\mathrm{v}_\mu^{(n)} s^{(n)} - i s^{(n)}\mathrm{v}_\mu^{(n+1)}}\right.$$

$$\boxed{\begin{array}{l} +i\left(\partial_\mu \mathrm{v}_5^{(n)} + i\mathrm{v}_\mu^{(n)}\mathrm{v}_5^{(n)} - i\mathrm{v}_5^{(n)}\mathrm{v}_\mu^{(n+1)}\right) \\ +i\left(\mathrm{v}_\mu^{(n)} - \mathrm{v}_\mu^{(n+1)}\right)/a\Big|^2 \end{array}}$$

$$\xrightarrow{a\to 0} \frac{1}{2\boxed{g_5^2}}\int dx_5 \boxed{\mathrm{Tr}\,(D_\mu s)^2} - \boxed{\mathrm{Tr}\,\mathrm{v}_{\mu 5}\mathrm{v}^{\mu 5}} + O(a)$$

$g_5^2 \equiv g^2 a$ (kept fixed as a-> 0)

AHCG action continued:

2. Scalar self-coupling “D”-term:

$$\frac{1}{2g^2}\sum_n \mathrm{Tr}\left(\phi^{(n+1)}\bar{\phi}^{(n+1)} - \bar{\phi}^{(n)}\phi^{(n)}\right)$$

$$\xrightarrow{a\to 0} \frac{1}{2g_5^2}\int dx_5 \mathrm{Tr}\,(D_5\, s)^2$$

AHCG action continued:

3. Fermion kinetic term

$$\frac{1}{g^2}\sum_n \mathrm{Tr}\left(\bar{\lambda}^{(n)} i\bar{\sigma}^\mu D_\mu \lambda^{(n)} + \bar{\psi}^{(n)} i\bar{\sigma}^\mu D_\mu \psi^{(n)}\right)$$

$$\xrightarrow{a\to 0} \frac{1}{g_5^2}\int dx_5\, \mathrm{Tr}\, \bar{\Psi} i\gamma^\mu D_\mu \Psi$$

$$\Psi = \begin{pmatrix} \lambda \\ \bar{\psi} \end{pmatrix}, \quad \bar{\Psi} = \begin{pmatrix} \psi & \bar{\lambda} \end{pmatrix}$$

$$\gamma_\mu = \begin{pmatrix} & \bar{\sigma}_\mu \\ \sigma_\mu & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

AHCG action continued:

4. Squark-quark-gaugino Yukawa coupling:

$$\frac{1}{g^2}\sum_n i\sqrt{2}\,\mathrm{Tr}\,\lambda^{(n)}\left(\psi^{(n)}\bar{\phi}^{(n)} - \bar{\phi}^{(n-1)}\psi^{(n-1)}\right) + \mathrm{h.c.}$$

$$\xrightarrow{a\to 0}\ \frac{1}{g_5^2}\int dx_5\,\mathrm{Tr}\,\lambda\,(iD_5\psi + [\psi, s]) + \mathrm{h.c} + O(a)$$

$$= \frac{1}{g_5^2}\int dx_5\,\mathrm{Tr}\,\bar{\Psi} i\gamma_5 D_5\Psi - \mathrm{Tr}\,\bar{\Psi}\gamma_5[s, \Psi]$$

After the $a \to 0$ limit, one finds a 5D theory:

- Gauge invariant
- 5D Lorentz invariant

$$v_m = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \lambda \\ \bar{\psi} \end{pmatrix}, \quad \bar{\Psi} = \begin{pmatrix} \psi & \bar{\lambda} \end{pmatrix} \qquad s$$

5D gauge field
v₅ from $Im[\phi]$

5D Dirac field
4 components
from λ, ψ

Real scalar s
from $Re[\phi]$

...and theory still respects the 4D supersymmetry.

What is amazing about the AHCG model

- 5D Lorentz invariant
- 4D supersymmetric = 2 complex or 4 real supercharges

But there is no 5D Lorentz invariant theory with 4 real supercharges!

Minimum number of supercharges = 8

In "IR" (a -> 0) we see:

- enhanced Lorentz symmetry
- enhanced SUSY

Magical!

Part V.

From orbifolds to lattices

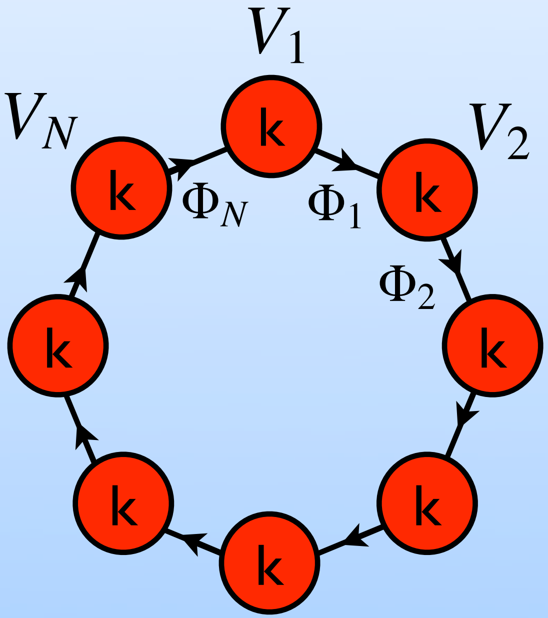
- A symmetry approach
- AHCG from orbifold projection
- Constructing SUSY lattices

V. Orbifolds to lattices

Goal:

- Find a general principle behind the AHCG model's successful realization of accidental SUSY in IR
- Harness that principle to construct true SUSY lattices.

Where did this come from?
What are its analogues for a full spacetime lattice?

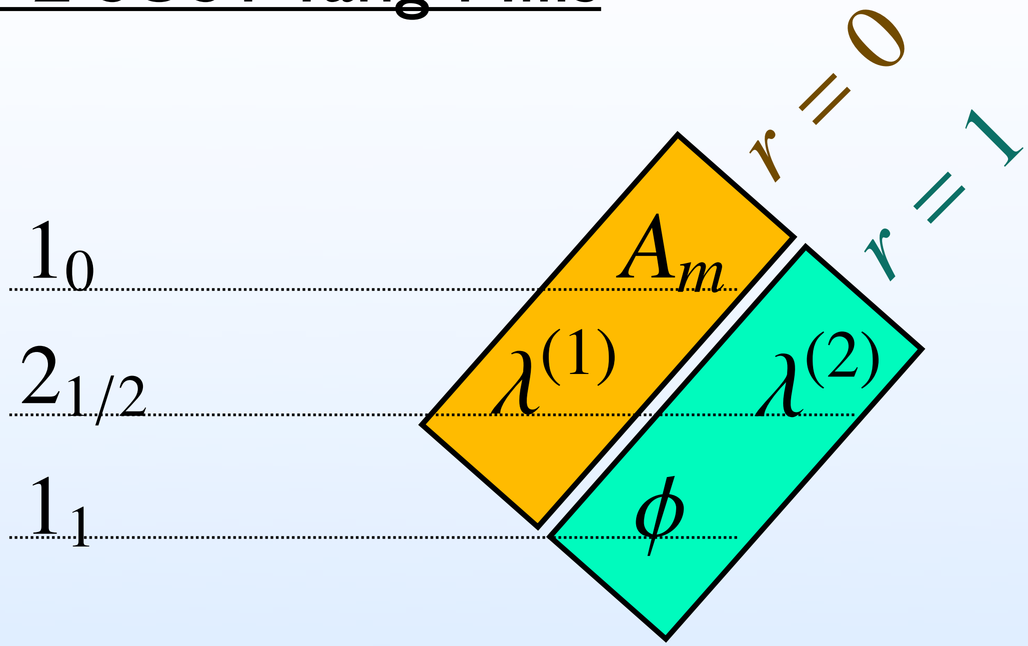


A symmetry approach to the AHCG model

Consider extended supersymmetric Yang-Mills theories in 4D:

Supersymmetry	# Real Super-charges	# gauge bosons	# Weyl gauginos	# complex scalars
	Q	v_m	λ	ϕ
N=1	4	1	1	0
N=2	8	1	2	1
N=4	16	1	4	3

N=2 SUSY Yang-Mills



gauge field

2 gauginos

1 complex scalar

Theory possesses an SU(2)xU(1) global "R"-symmetry

Define charge $$r = \underset{\text{U(1)}}{\underset{\nearrow}{Y}} - \underset{\text{SU(2)}}{\underset{\nwarrow}{T_3}}$$

r=0 -> sites and r=1 -> links in AHCG model

Orbifolds

How to get the AHCG model from an *N*=2 SYM theory through symmetry projection

1. Start with an N=2 supersymmetric gauge theory with gauge group U(kN)

All fields are adjoints: kN x kN matrices

2. Define a Z_N subgroup of $\underbrace{U(kN)}_{\textbf{gauge}}\times\underbrace{SU(2)\times U(1)}_{\textbf{R}}$

$r = (Y - T_3)$

$$\hat{\gamma} = \omega^r \begin{pmatrix} \omega & & & \\ & \omega^2 & & \\ & & \ddots & \\ & & & \omega^N \end{pmatrix} \qquad \omega = e^{2\pi i/N}$$

Action of Z_N generator on a field X in the N=2 SUSY theory:

$$\boxed{\hat{\gamma} X = \omega^r \Omega X \Omega^\dagger}$$

Projection operator which annihilates anything transforming nontrivially under the Z_N:

$$\boxed{\hat{P} = \frac{1}{N} \sum_{n=1}^{N} \hat{\gamma}^n}$$

$\hat{P}$ ="orbifold projection operator"

3. Project out of the theory all variables that are charged under the Z_N. (Note different variables have different **r** charges):

$$\boxed{X \to \hat{P}X = \omega^r \Omega(\hat{P}X)\Omega^\dagger}$$

- Only N kxk blocks survive in the original kN x kN matrix variable X
- Which blocks survive depends on the r-charge of X

i. **r=0**: diagonal blocks survive.

Interpreted as **site variables** on N site lattice

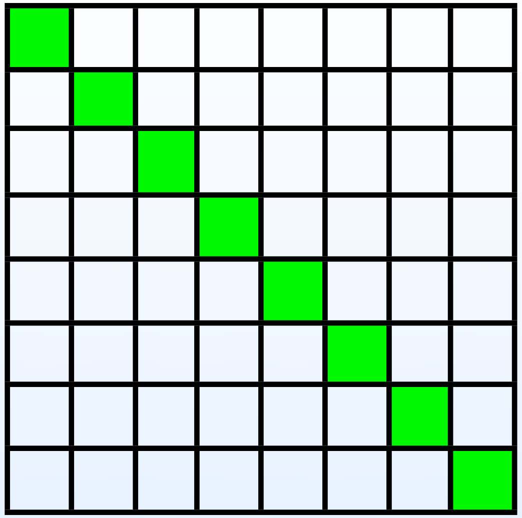
ii. **r=1**: super-diagonal blocks survive.

Interpreted as **link variables** on N-site lattice

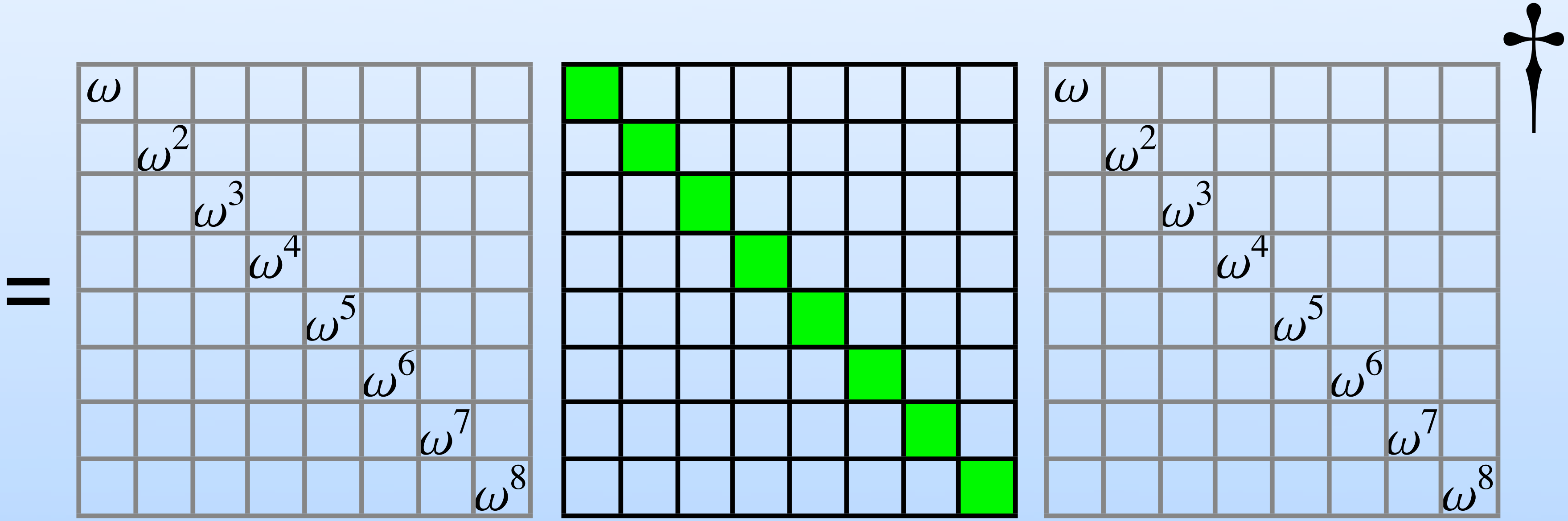
Example: gauge group = $U(8k)$

Project out Z_8 $\left(\omega = e^{2\pi i/8}\right)$

$r = 0$ variables become kxk site variables on 8-site, 1d lattice ($X = \mathrm{v_m}$ and $X = \lambda_1$)



$\hat{P}X$



$$= \Omega(\hat{P}X)\Omega^\dagger$$

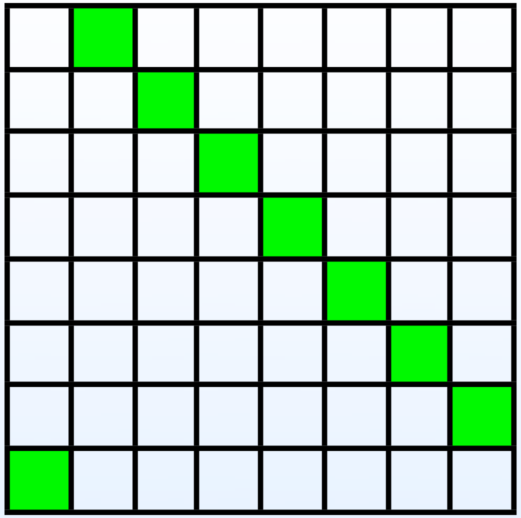
From orbifolds to lattices

A symmetry approach

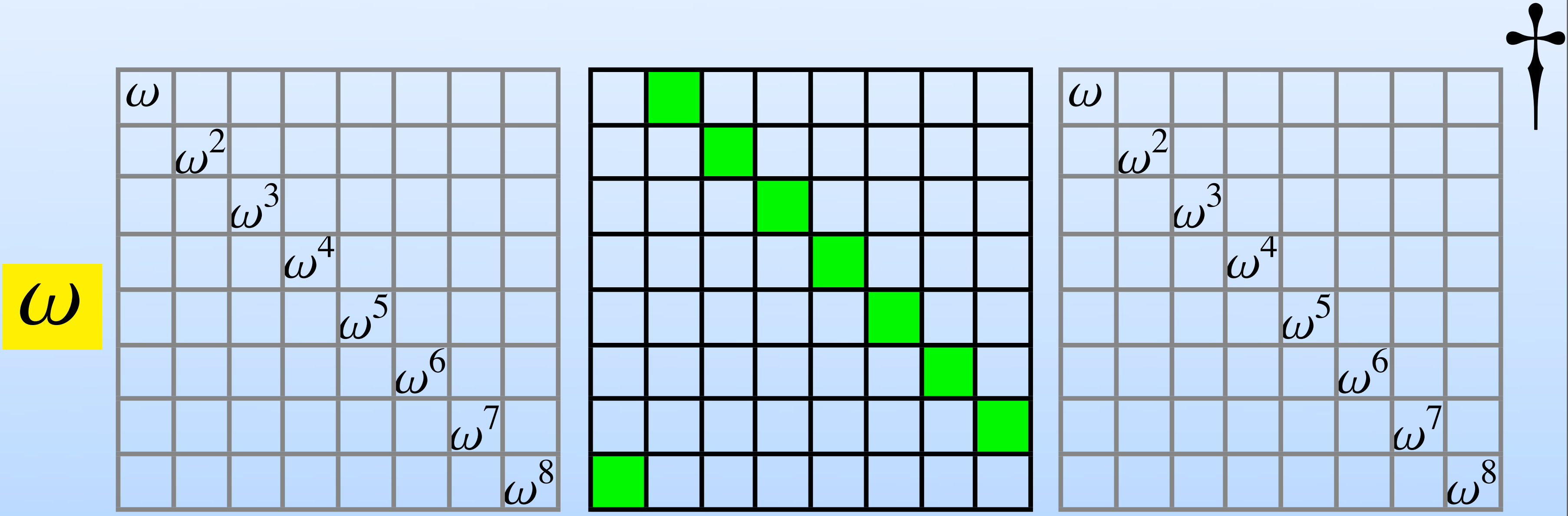
AHCG from orbifold projection

Constructing SUSY lattices

$r = 1$ variables become kxk link variables on 8-site, 1d lattice ($X = \phi$ and $X = \lambda_2$)



$\hat{P}X$



$$= \omega\, \Omega(\hat{P}X)\Omega^\dagger$$

After the projection, compute the original action (N=2 SYM) with the sparse matrix variables. One gets the AHCG model.

Symmetry of original action has been reduced:

- ✷ gauge symmetry: $U(kN) \Rightarrow U(k)^N$
- ✷ supersymmetry: N=2 (8Q's) ⇒ N=1 (4Q's)

Deconstruction procedure then restores the broken Q's and adds a dimension!

4d, *N*=2 SYM (8 real Q's)
U(kN) gauge symmetry

Z_N "orbifold" projection

AHCG model
4d *N*=1 SYM (4 real Q's)
$U(k)^N$ gauge symmetry

Deconstruction limit

$\langle \phi \rangle \propto 1/a$

$a \to 0,\ N \to \infty$

5d *N*=1 SYM (8 real Q's)
U(k) gauge symmetry

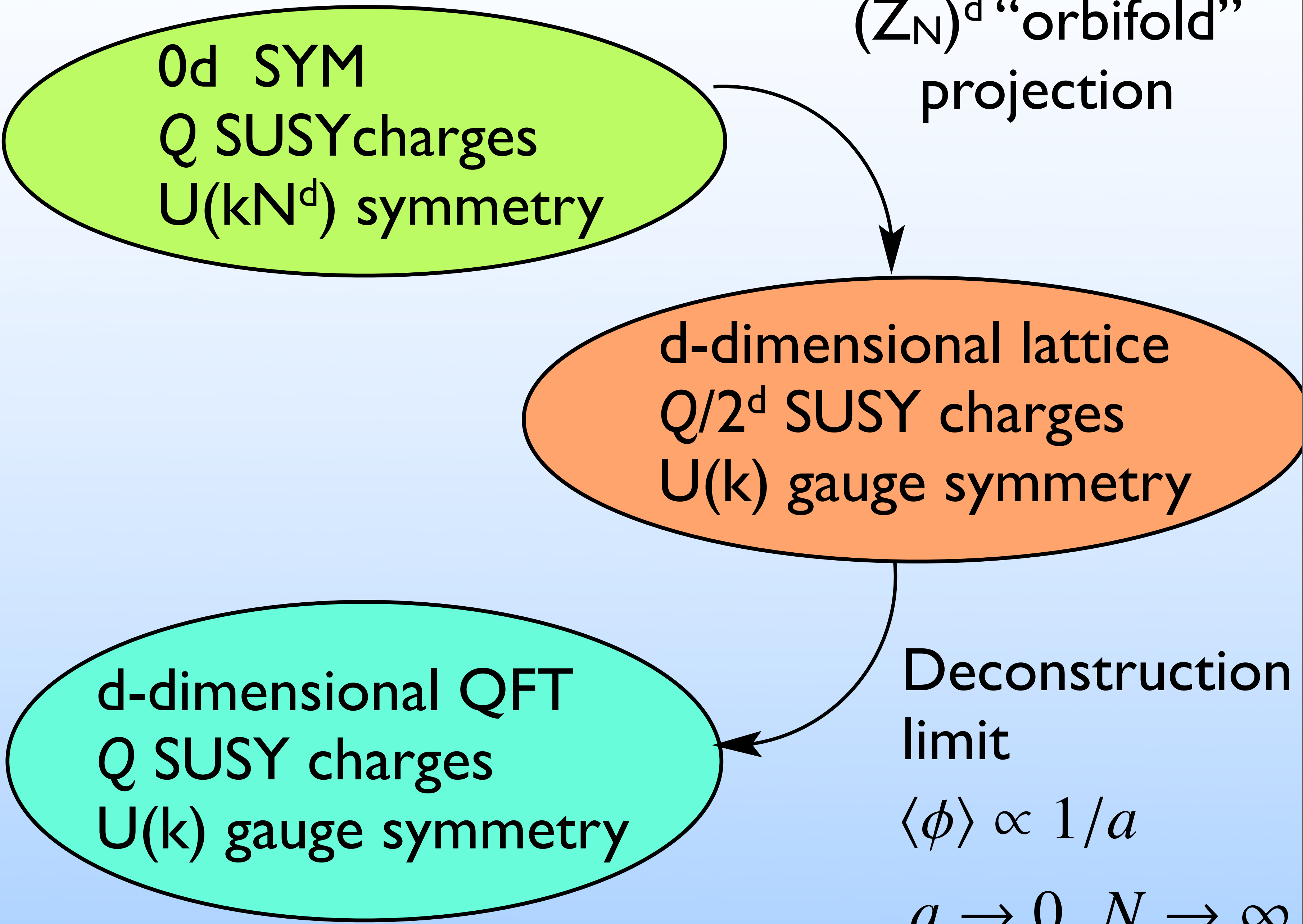
From orbifolds to lattices

A symmetry approach

AHCG from orbifold projection

Constructing SUSY lattices

Application to SUSY lattices



0d SYM

Q SUSYcharges

U(kN^d) symmetry

(Z_N)^d "orbifold" projection

d-dimensional lattice

Q/2^d SUSY charges

U(k) gauge symmetry

Deconstruction limit

$\langle \phi \rangle \propto 1/a$

$a \to 0, \; N \to \infty$

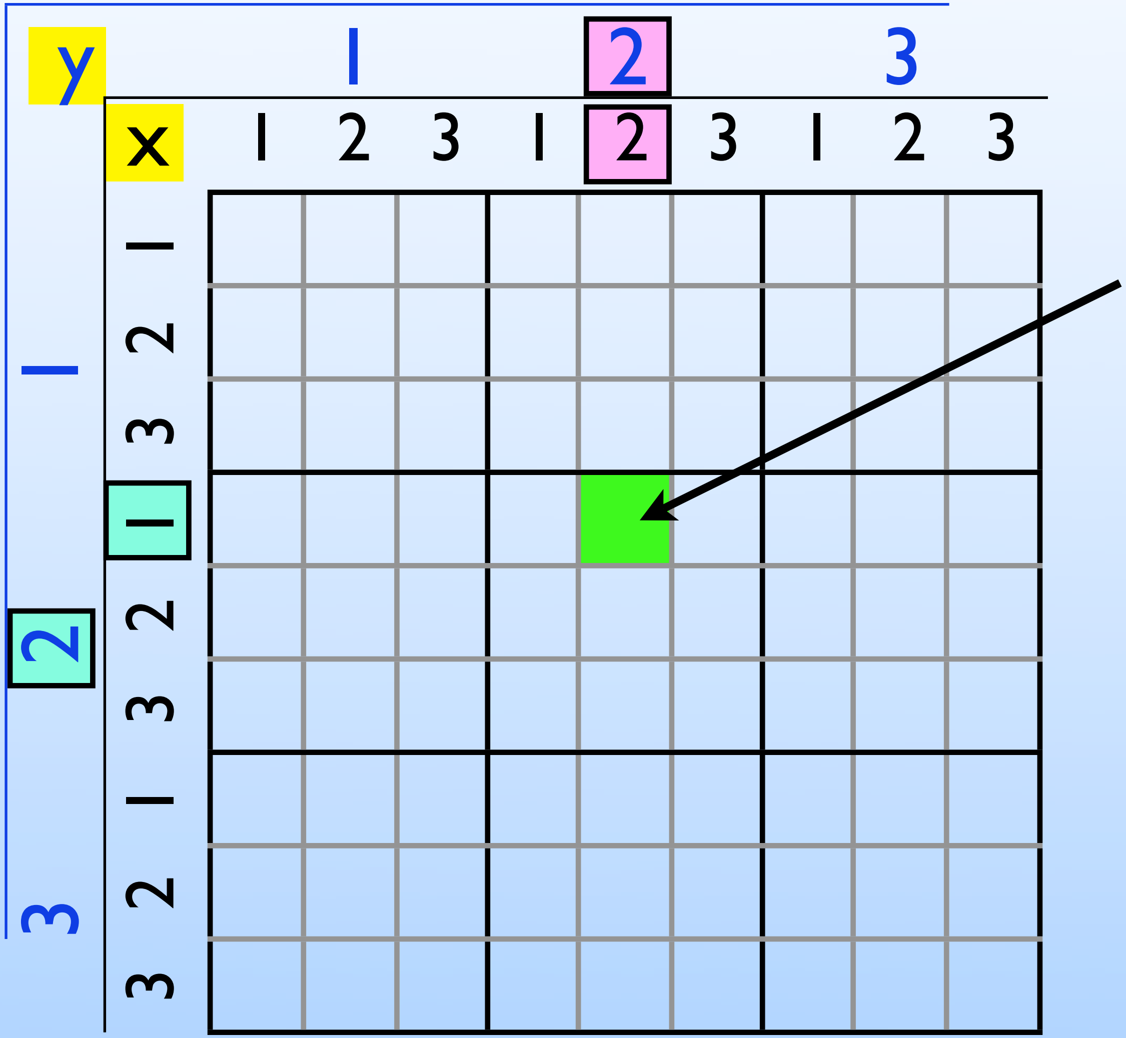
d-dimensional QFT

Q SUSY charges

U(k) gauge symmetry

Example: 2d lattice from Z_N^2 orbifold

Encoding link & site variables on a 2d lattice in a matrix (here, 3x3 lattice in 9x9 matrix)



Matrix-valued link variable

from (x,y)=(1,2)

to (x,y)=(2,2)

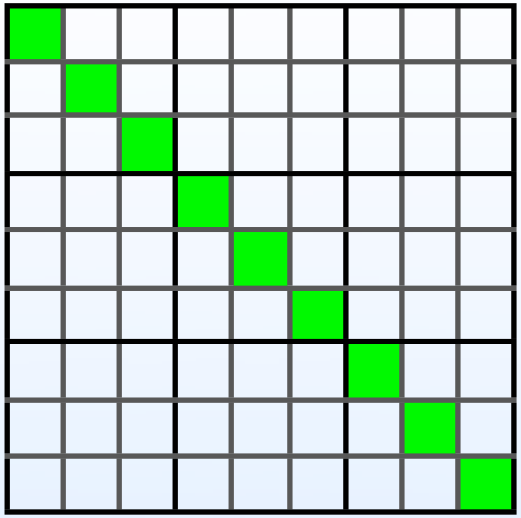
Generating a 2d lattice from a $Z_N{}^2$ orbifold projection

$Z_N{}^2$ generators:

$$\gamma_1 = \omega^{r_1}\Omega_1\ , \qquad \gamma_2 = \omega^{r_2}\Omega_2\ ,$$

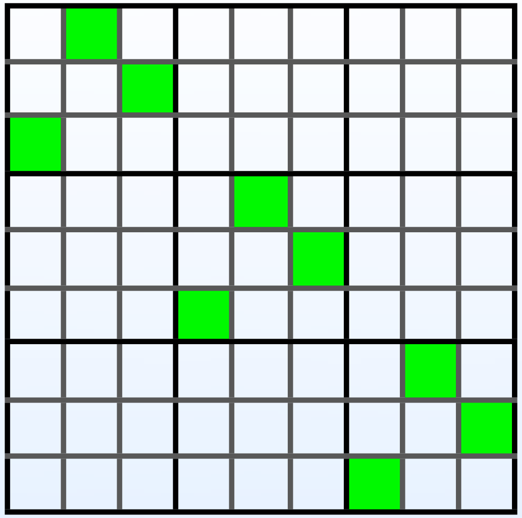
$$\Omega_1 = \left(\begin{array}{ccc|ccc|ccc}
\omega & & & & & & & & \\
 & \omega^2 & & & & & & & \\
 & & \ddots & & & & & & \\
\hline
 & & & \omega & & & & & \\
 & & & & \omega^2 & & & & \\
 & & & & & \ddots & & & \\
\hline
 & & & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & & & & & & & &
\end{array}\right) \qquad
\Omega_2 = \left(\begin{array}{ccc|ccc|ccc}
\omega & & & & & & & & \\
 & \omega & & & & & & & \\
 & & \ddots & & & & & & \\
\hline
 & & & \omega^2 & & & & & \\
 & & & & \omega^2 & & & & \\
 & & & & & \ddots & & & \\
\hline
 & & & & & & \ddots & & \\
 & & & & & & & \ddots & \\
 & & & & & & & &
\end{array}\right)$$

$(r_1, r_2) = (0, 0)$



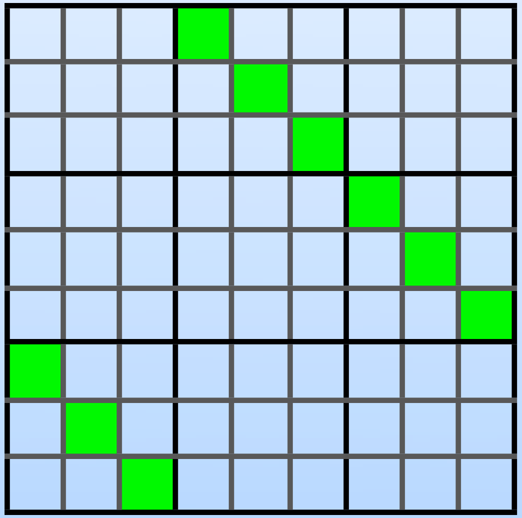
site variables

$(r_1, r_2) = (1, 0)$



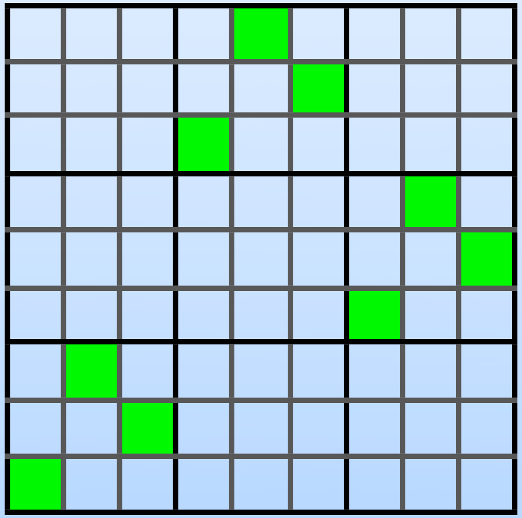
x-link variables

$(r_1, r_2) = (0, 1)$



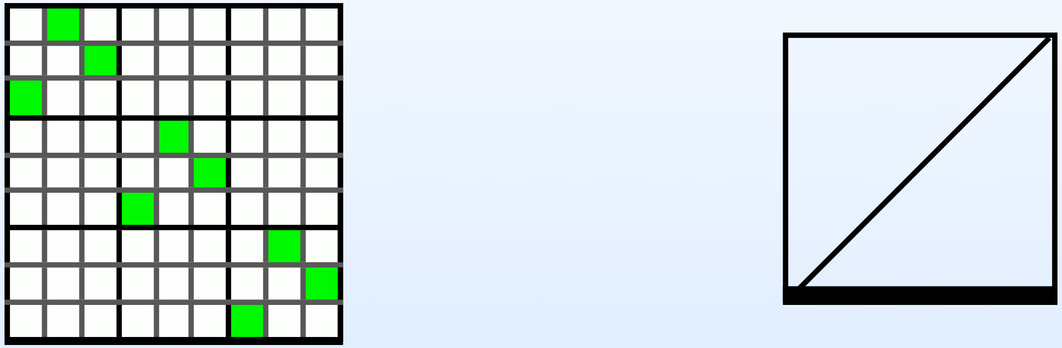
y-link variables

$(r_1, r_2) = (1, 1)$

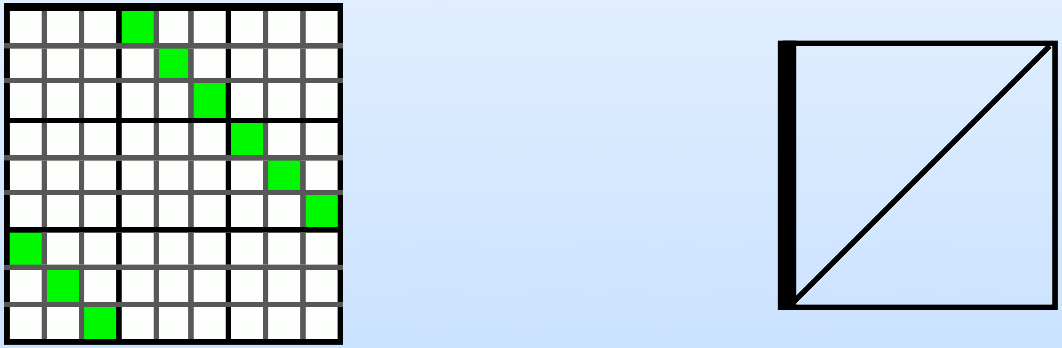


diagonal-link variables

$$(r_1, r_2) = (0, 0)$$



$$(r_1, r_2) = (1, 0)$$



$$(r_1, r_2) = (0, 1)$$

$$(r_1, r_2) = (1, 1)$$

Summary:

By “sparsifying” adjoint representations of $U(kN^d)$, into N^d k x k blocks, we can turn the internal group space into a physical d-dimensional N^d site lattice with a U(k) gauge symmetry.
Matrix commutators turn into derivatives.

“Sparsifying” can be accomplished by projecting out a Z_N^d symmetry of the theory (orbifolding)

In a SUSY YM theory, if the Z_N^d symmetry is properly embedded in the gauge x R-symmetry, the resultant lattice will enjoy residual SUSY.

A continuum limit can be defined which at the classical level restores all of the original theory's SUSY, as well as d-dimensional Lorentz symmetry

Tomorrow: construct the 2d lattice for (2,2) SUSY YM (4 supercharges in 2d), and explore the renormalization properties.

Part VI.

A lattice for (2,2) SUSY YM

- The target theory
- Constructing the lattice
- The lattice action
- Dispersion relations
- Lattice SUSY
- Radiative corrections
- Other theories

Constructing a supersymmetric lattice

Target theory: (2,2) Super Yang-Mills in 2d

Continuum action obtained by reducing N=1 SYM from 4d to 2d

- *4d gauge field* ⇨ *2d gauge field + complex scalar*
- *4d Weyl fermion* ⇨ *2d Dirac fermion*

$$\mathcal{L} = \frac{1}{g_2^2}\mathrm{Tr}\Big(|D_m s|^2 + i\bar{\psi}\,\not{D}\psi + \tfrac{1}{4}v_{mn}v_{mn} \\ +i\sqrt{2}\left(\bar{\psi}_L[s,\psi_R] + \bar{\psi}_R[s^\dagger,\psi_L]\right) + \tfrac{1}{2}[s^\dagger,s]^2\Big)$$

Orbifold method for a 2d SUSY lattice:

(1) Start in zero dimensions with an action invariant under a $U(kN^2)$ gauge symmetry and 4 supercharges

(2) Project out a $Z_N \times Z_N$ symmetry

(3) Identify the "flat direction" of the theory (moduli space) and equate a scalar vev with an inverse lattice spacing

(4) Take the appropriate continuum limit

Step (1): Creating a zero dimensional “mother theory”

- Start with N=1 U(kN^2) SYM in 4d
- Reduce to zero dimensions

Result (just erase all spacetime dependence in gluon/gluino fields!) is a matrix model in zero dimensions:

$$\mathcal{L}_0 = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}\mathrm{v}_{mn}\mathrm{v}_{mn} + \bar{\psi}\bar{\sigma}_m[\mathrm{v}_m,\psi]\right)$$

$$\mathrm{v}_{mn} \equiv i[\mathrm{v}_m,\mathrm{v}_n] \qquad m,n = 1,\ldots,4$$

Still possesses all 4 supercharges

Step (2): Identify symmetries

$$\mathcal{L}_0 = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}\mathrm{v}_{mn}\mathrm{v}_{mn} + \bar{\psi}\bar{\sigma}_m[\mathrm{v}_m, \psi]\right)$$

- U(kN^2): $\mathrm{v}_m \to U\mathrm{v}_m U^\dagger, \quad \psi \to U\psi U^\dagger$

$$\left.\begin{array}{ll}\bullet\ \text{SO(4):} & \text{“Lorentz” transformation}\\[2ex] \bullet\ \text{U(1):} & \psi \to e^{i\alpha}\psi\end{array}\right\}\ R$$

- SUSY:

$$\delta = i\kappa Q + i\bar{\kappa}\bar{Q}$$

$\kappa,\ \bar{\kappa}$ = 2 component **Grassmann** spinor parameters

$$\begin{aligned}\delta\mathrm{v}_m &= -i\bar{\psi}\,\bar{\sigma}_m\kappa + i\bar{\kappa}\,\bar{\sigma}_m\psi\\ \delta\psi &= -i\mathrm{v}_{mn}\sigma_{mn}\kappa\\ \delta\bar{\psi} &= i\mathrm{v}_{mn}\,\bar{\kappa}\,\bar{\sigma}_{mn}\end{aligned}$$

Step (3): Identify $Z_N \times Z_N$ charges for orbifold

$$\hat{\gamma}_1 = \omega^{r_1}\Omega_1\ , \qquad \hat{\gamma}_2 = \omega^{r_2}\Omega_2\ ,$$

$$\hat{P}_{\rm 2d} = \frac{1}{N^2}\sum_{n,m=1}^{N} \hat{\gamma}_1^m \hat{\gamma}_2^n$$

- r_1, r_2 constructed from diagonal generators of the SO(4)xU(1) R-symmetry (rank 3).
- Maximize # of preserved supercharges = number of fermions with $(r_1,r_2)=(0,0)$
- Only allow values 0, +1, -1 for the $r_{1,2}$ (near neighbor interactions only)

A solution:

(Cohen, Kaplan, Katz, Unsal, hep-lat/0302017)

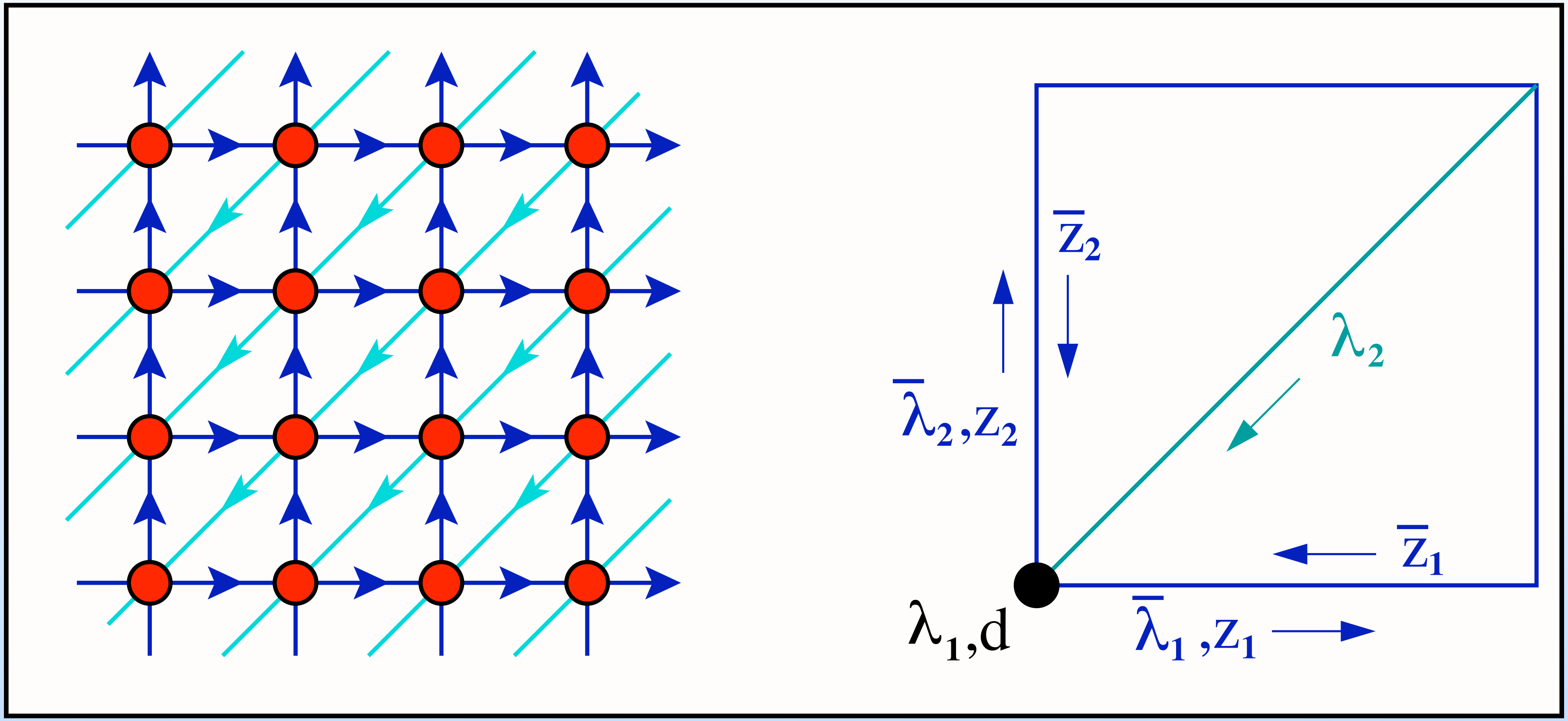
	r_1	r_2
$z_1 = \frac{1}{\sqrt{2}}(\mathrm{v}_0 - i\mathrm{v}_3)$	1	0
$\bar{z}_1 = \frac{1}{\sqrt{2}}(\mathrm{v}_0 + i\mathrm{v}_3)$	-1	0
$z_2 = -\frac{i}{\sqrt{2}}(\mathrm{v}_1 - i\mathrm{v}_2)$	0	1
$\bar{z}_2 = \frac{i}{\sqrt{2}}(\mathrm{v}_1 + i\mathrm{v}_2)$	0	-1

	r_1	r_2
λ_1	0	0
λ_2	-1	-1
$\bar{\lambda}_1$	1	0
$\bar{\lambda}_2$	0	1

$$\psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \end{pmatrix}$$

Only one supercharge survives orbifold projection

The “lattice”:

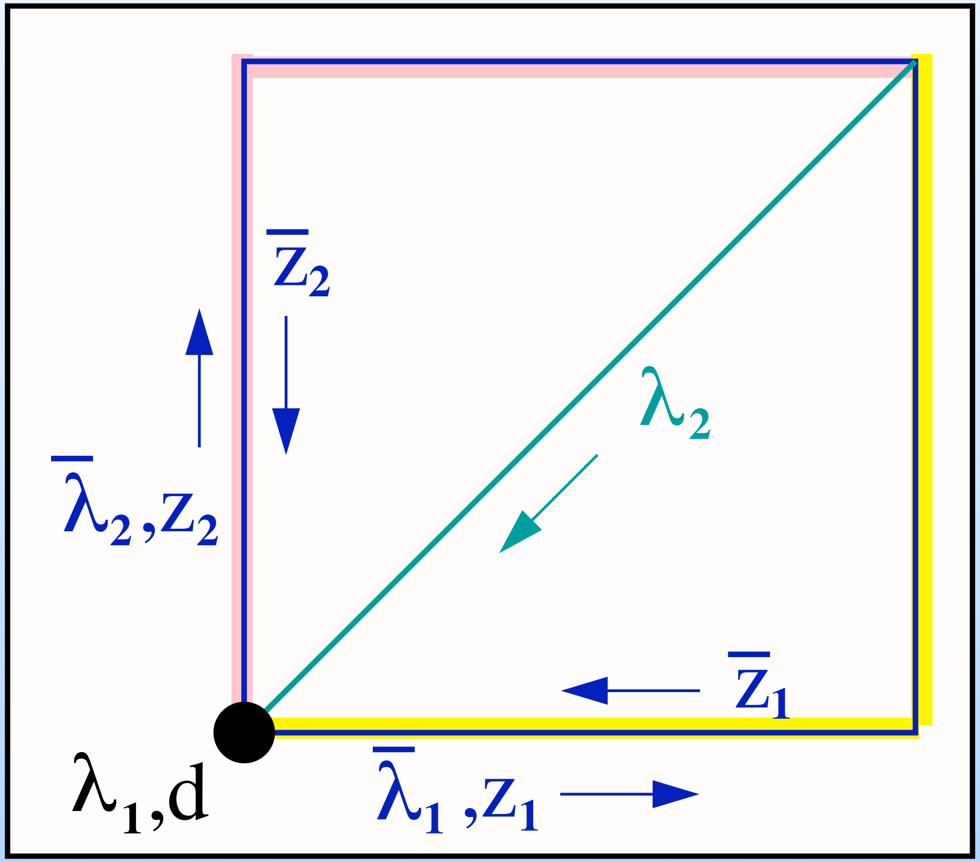


1. Stick the sparse, projected matrices into the action of the mother theory:

$$\mathcal{L}_0 = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}v_{mn}v_{mn} + \bar{\psi}\bar{\sigma}_m[v_m,\psi]\right)$$

Arrive at the action $S=S_{\mathrm{bose}}+S_{\mathrm{fermi}}$

$$S_{\text{bose}} = \frac{1}{g^2}\sum_{\mathbf{n}} \mathrm{Tr}\Big[\tfrac{1}{2}\big(\bar{z}_1(\mathbf{n}-\hat{\mathbf{x}})z_1(\mathbf{n}-\hat{\mathbf{x}}) - z_1(\mathbf{n})\bar{z}_1(\mathbf{n}) + \bar{z}_2(\mathbf{n}-\hat{\mathbf{y}})z_2(\mathbf{n}-\hat{\mathbf{y}}) - z_2(\mathbf{n})\bar{z}_2(\mathbf{n})\big)^2 + 2\left|\boxed{z_1(\mathbf{n})z_2(\mathbf{n}+\hat{\mathbf{x}})} - \boxed{z_2(\mathbf{n})z_1(\mathbf{n}+\hat{\mathbf{y}})}\right|^2\Big]$$



A 2d example

The target

Lattice construction

The lattice action

Dispersion relations

Lattice SUSY

Radiative corrections

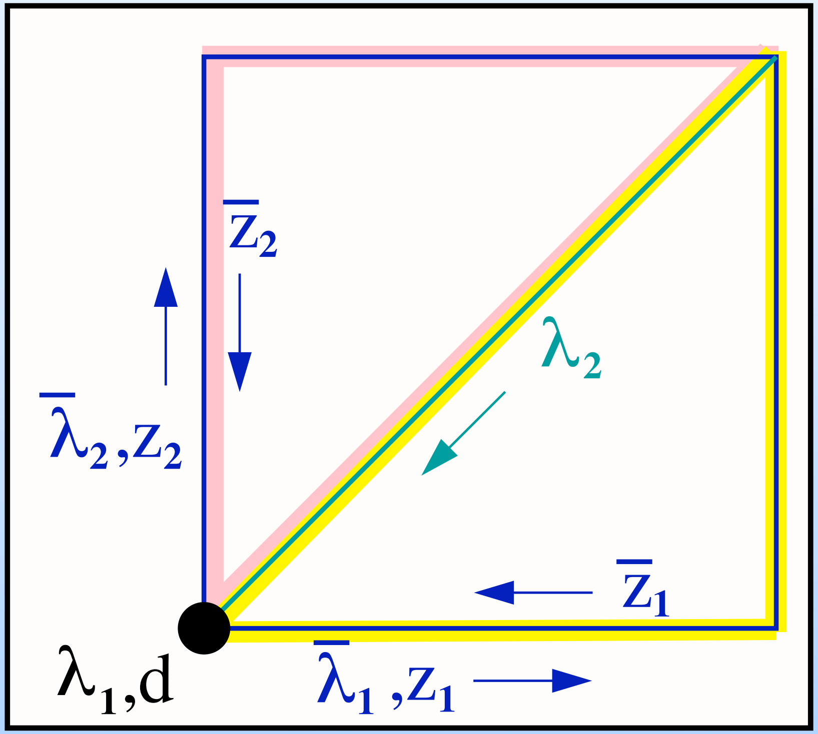
Other theories

$$S_{\text{fermi}} = \frac{1}{g^2}\sum_{\mathbf{n}} \text{Tr}\Big[\sqrt{2}\,\Big(\bar\lambda_1(\mathbf{n})\bar z_1(\mathbf{n})\lambda_1(\mathbf{n}) - \bar\lambda_1(\mathbf{n}-\hat{\mathbf{x}})\lambda_1(\mathbf{n})\bar z_1(\mathbf{n}-\hat{\mathbf{x}})\Big)$$

$$+\sqrt{2}\,\Big(\bar\lambda_2(\mathbf{n})\bar z_2(\mathbf{n})\lambda_1(\mathbf{n}) - \bar\lambda_2(\mathbf{n}-\hat{\mathbf{y}})\lambda_1(\mathbf{n})\bar z_2(\mathbf{n}-\hat{\mathbf{y}})\Big)$$

$$\boxed{-\sqrt{2}\,\Big(\bar\lambda_1(\mathbf{n})z_2(\mathbf{n}+\hat{\mathbf{x}})\lambda_2(\mathbf{n})}\boxed{-\bar\lambda_1(\mathbf{n}+\hat{\mathbf{y}})\lambda_2(\mathbf{n})z_2(\mathbf{n})\Big)}$$

$$+\sqrt{2}\,\Big(\bar\lambda_2(\mathbf{n})z_1(\mathbf{n}+\hat{\mathbf{y}})\lambda_2(\mathbf{n}) - \bar\lambda_2(\mathbf{n}+\hat{\mathbf{x}})\lambda_2(\mathbf{n})z_1(\mathbf{n})\Big)\Big]$$

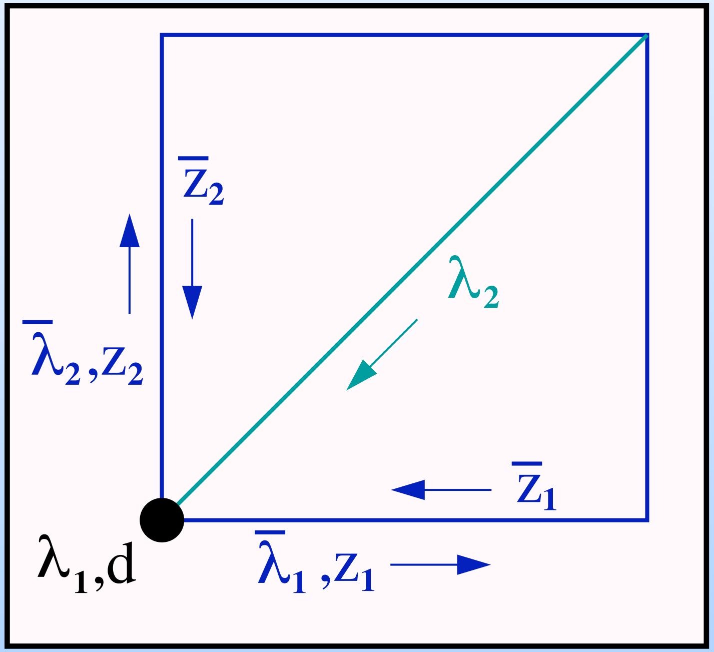


2. Theory has a moduli space (=degenerate vacua) parametrized by

$$\langle z_1(\mathbf{n})\rangle = \langle z_2(\mathbf{n})\rangle = \frac{1}{\sqrt{2}\boxed{a}}\mathbf{1}_k$$

k x k unit matrix

to become the lattice spacing



Expand fields about this point

$$\langle z_1(\mathbf{n})\rangle = \langle z_2(\mathbf{n})\rangle = \frac{1}{\sqrt{2}\,a}\,\mathbf{1}_k$$

Quadratic part of the fermionic action:

$$\frac{1}{g^2}\sum_n \frac{1}{a}\mathrm{Tr}\Big[\Big(\bar\lambda_1(\mathbf{n})-\bar\lambda_1(\mathbf{n}-\hat{\mathbf{x}})\Big)\lambda_1(\mathbf{n}) + \Big(\bar\lambda_2(\mathbf{n})-\bar\lambda_2(\mathbf{n}-\hat{\mathbf{y}})\Big)\lambda_1(\mathbf{n})$$

$$- \Big(\bar\lambda_1(\mathbf{n})-\bar\lambda_1(\mathbf{n}+\hat{\mathbf{y}})\Big)\lambda_2(\mathbf{n}) + \Big(\bar\lambda_2(\mathbf{n})-\bar\lambda_2(\mathbf{n}+\hat{\mathbf{x}})\Big)\lambda_2(\mathbf{n})\Big] + O(a)$$

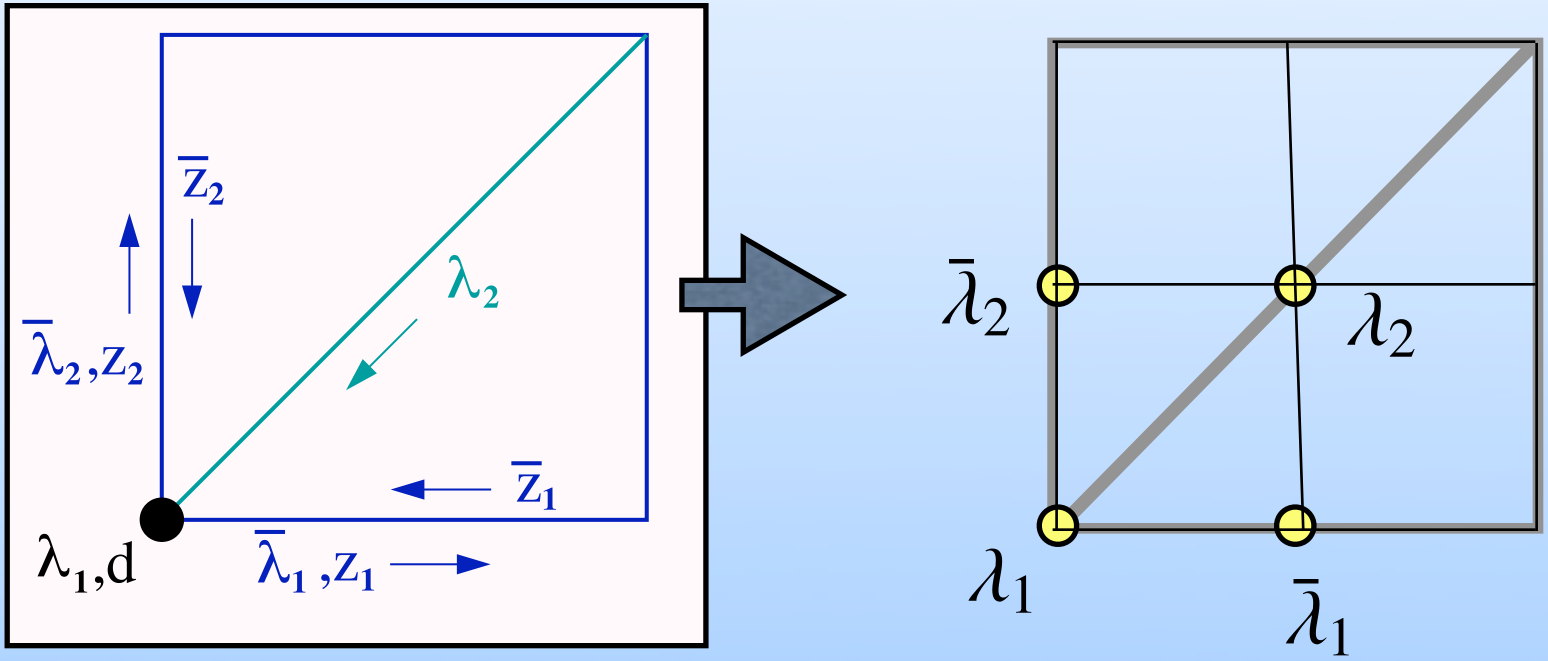
$$= \frac{1}{g^2}\sum_{\mathbf{p}} \begin{pmatrix}\bar\lambda_1(\mathbf{p}) & \bar\lambda_2(\mathbf{p})\end{pmatrix} \boxed{iK(\mathbf{p})} \begin{pmatrix}\lambda_1(-\mathbf{p}) \\ \lambda_2(-\mathbf{p})\end{pmatrix}$$

$$\boxed{K^\dagger K} = \left(\mathcal{P}_x^2 + \mathcal{P}_y^2\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \mathcal{P}_i \equiv \frac{2}{a}\sin\frac{ap_i}{2}$$

$\mathcal{P}_i\Big|_{p_i=\pm\pi/a} \neq 0 \Longleftrightarrow$ No doublers.

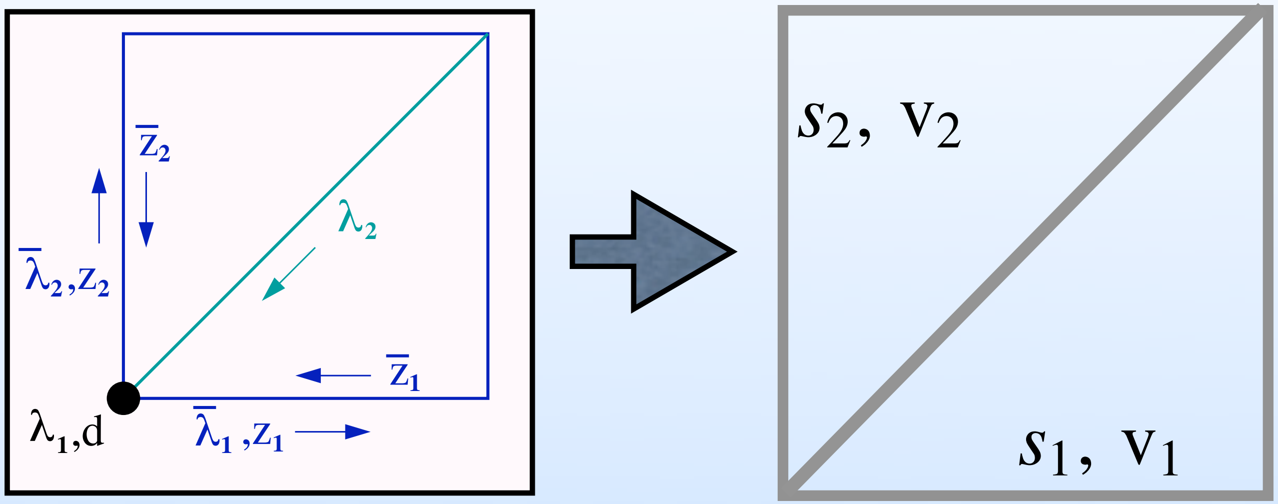
Have we invented a new type of fermion??

No, these are staggered fermions in disguise (“reduced staggered fermions”) on a lattice with spacing a/2



What about the bosons? Staggered scalars?

$$z_i(\mathbf{n}) = \frac{1}{\sqrt{2}}\left(\frac{1}{a}\,\mathbf{1}_k + s_i(\mathbf{n}) + i\,\mathrm{v}_i(\mathbf{n})\right)$$



$$\frac{1}{2g^2a^2}\sum_{\mathbf{n}}\mathrm{Tr}\Big[\left(s_{1,\mathbf{n}-\hat{\mathbf{x}}} - s_{1,\mathbf{n}} + s_{2,\mathbf{n}-\hat{\mathbf{y}}} - s_{2,\mathbf{n}}\right)^2 + \left|\left(s_{1,\mathbf{n}+\hat{\mathbf{y}}} - s_{1,\mathbf{n}} + s_{2,\mathbf{n}} - s_{2,\mathbf{n}+\hat{\mathbf{x}}}\right) - i\left(\mathrm{v}_{1,\mathbf{n}+\hat{\mathbf{y}}} - \mathrm{v}_{1,\mathbf{n}} - \mathrm{v}_{2,\mathbf{n}+\hat{\mathbf{x}}} + \mathrm{v}_{2,\mathbf{n}}\right)\right|^2\Big]$$

Bosonic action at quadratic order

$$\to (\partial_1 s_1 + \partial_2 s_2)^2$$

$$\to (\partial_2 s_1 - \partial_1 s_2)^2$$

$$\frac{1}{2g^2a^2}\sum_{\mathbf{n}}\mathrm{Tr}\Big[\big(s_{1,\mathbf{n}-\hat{\mathbf{x}}} - s_{1,\mathbf{n}} + s_{2,\mathbf{n}-\hat{\mathbf{y}}} - s_{2,\mathbf{n}}\big)^2 + \Big|\big(s_{1,\mathbf{n}+\hat{\mathbf{y}}} - s_{1,\mathbf{n}} + s_{2,\mathbf{n}} - s_{2,\mathbf{n}+\hat{\mathbf{x}}}\big) - i\big(v_{1,\mathbf{n}+\hat{\mathbf{y}}} - v_{1,\mathbf{n}} - v_{2,\mathbf{n}+\hat{\mathbf{x}}} + v_{2,\mathbf{n}}\big)\Big|^2\Big]$$

$$= \frac{1}{2g^2}\sum_{\mathbf{n}}\mathrm{Tr}\left[\sum_{\hat{\mu}}\sum_{i=1,2}\left(\frac{s_{i,\mathbf{n}} - s_{i,\mathbf{n}-\hat{\mu}}}{a}\right)^2 + \left(\frac{v_{1,\mathbf{n}+\hat{\mathbf{y}}} - v_{1,\mathbf{n}}}{a} - \frac{v_{2,\mathbf{n}+\hat{\mathbf{x}}} - v_{2,\mathbf{n}}}{a}\right)^2\right]$$

$$\to (\partial_m s_1)^2 + (\partial_m s_2)^2$$

$$\to (\partial_2 v_1 - \partial_1 v_2)^2$$

Bosonic dispersion relations:

Complex scalar $s = \dfrac{s_1 + i s_2}{\sqrt{2}}$

$$\sum_{\mathbf{p}} s^*(\mathbf{p}) \left[\mathcal{P}_x^2 + \mathcal{P}_y^2\right] s(-\mathbf{p}) \qquad \mathcal{P}_i \equiv \tfrac{2}{a} \sin \tfrac{a p_i}{2}$$

Same as fermions!

Gauge bosons: $\displaystyle\sum_{\mathbf{p},m,n} \mathrm{v}_m(\mathbf{p})\, G(\mathbf{p})_{mn}\, \mathrm{v}_n(-\mathbf{p})$

$$G(\mathbf{p}) = \begin{pmatrix} \mathcal{P}_y^2 & -e^{-ia(p_x - p_y)}\mathcal{P}_x\mathcal{P}_y \\ -e^{ia(p_x-p_y)}\mathcal{P}_x\mathcal{P}_y & \mathcal{P}_x^2 \end{pmatrix} \xrightarrow{a\to 0} (\mathbf{p}^2 \delta_{ij} - p_i p_j)$$

Standard dispersion without gauge-fixing

Since no doublers, continuum limit is trivial to take

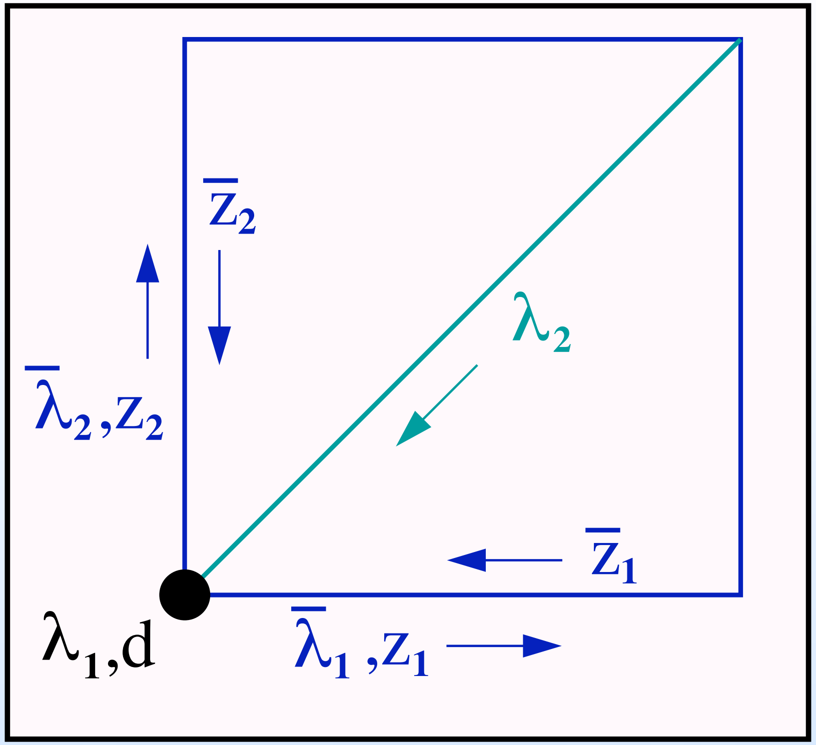
$$a \to 0\,,\;\; N \to \infty\,,\;\; g/a \to g_2\ (\text{fixed})\,,\;\; Na \to L\ (\text{fixed})$$

...and one finds the desired target theory

$$\begin{aligned}\mathcal{L} = \frac{1}{g_2^2}\mathrm{Tr}\Big(&|D_m s|^2 + i\bar{\psi}\,\not{D}\psi + \tfrac{1}{4}\mathrm{v}_{mn}\mathrm{v}_{mn}\\ &+i\sqrt{2}\left(\bar{\psi}_L[s,\psi_R] + \bar{\psi}_R[s^\dagger,\psi_L]\right) + \tfrac{1}{2}[s^\dagger,s]^2\Big)\end{aligned}$$

Is there any SUSY left on the lattice?

Yes! One supercharge Q.



$$\delta = i\eta Q$$

η = **Grassmann** parameter

$$\delta z_i(\mathbf{n}) = i\sqrt{2}\,\eta\bar{\lambda}_i(\mathbf{n})$$

$$\delta\lambda_1(\mathbf{n}) = -i\Big[\bar{z}_1(\mathbf{n}-\hat{\mathbf{x}})z_1(\mathbf{n}-\hat{\mathbf{x}}) - z_1(\mathbf{n})\bar{z}_1(\mathbf{n}) + \bar{z}_2(\mathbf{n}-\hat{\mathbf{y}})z_2(\mathbf{n}-\hat{\mathbf{y}}) - z_2(\mathbf{n})\bar{z}_2(\mathbf{n}) + id(\mathbf{n})\Big]\eta$$

$$\delta\lambda_2(\mathbf{n}) = 2i\Big[\bar{z}_1(\mathbf{n}+\hat{\mathbf{y}})\bar{z}_2(\mathbf{n}) - \bar{z}_2(\mathbf{n}+\hat{\mathbf{x}})\bar{z}_1(\mathbf{n})\Big]\eta$$

$$\delta\bar{z}_i(\mathbf{n}) = 0$$

$$\delta\bar{\lambda}_i(\mathbf{n}) = 0$$

(before shifting vev)

Analyzing theory is simplified by introducing superfields in terms of **Grassmann** coordinate θ, with $Q = \partial/\partial\theta$

$$
\begin{aligned}
\mathbf{Z_{1n}} &= z_1(\mathbf{n}) + \sqrt{2}\,\theta\,\bar{\lambda}_1(\mathbf{n}) \\
\mathbf{Z_{2n}} &= z_2(\mathbf{n}) + \sqrt{2}\,\theta\,\bar{\lambda}_2(\mathbf{n}) \\
\mathbf{\Lambda_n} &= \lambda_1(\mathbf{n}) \\
&\quad - \Big[\bar{z}_1(\mathbf{n}-\hat{\mathbf{x}})z_1(\mathbf{n}-\hat{\mathbf{x}}) - z_1(\mathbf{n})\bar{z}_1(\mathbf{n}) \\
&\qquad + \bar{z}_2(\mathbf{n}-\hat{\mathbf{y}})z_2(\mathbf{n}-\hat{\mathbf{y}}) - z_2(\mathbf{n})\bar{z}_2(\mathbf{n}) + id(\mathbf{n}))\Big]\theta \\
\mathbf{\Xi_n} &= \xi_{\mathbf{n}} + 2\left(\bar{z}_1(\mathbf{n}+\hat{\mathbf{y}})\bar{z}_2(\mathbf{n}) - \bar{z}_2(\mathbf{n}+\hat{\mathbf{x}})\bar{z}_1(\mathbf{n})\right)\,\theta
\end{aligned}
$$

The lattice action may be written in manifestly supersymmetric form using these superfields.

Have found so far:

- Can construct a 2d lattice action which reproduces the (2,2) SUSY YM action in the continuum limit at tree level *(including the desired U(1) chiral R-symmetry)*
- The lattice action possesses one exact supercharge, which can be made manifest by writing in terms of superfields
- Fermions are realized as "reduced staggered fermions" : one Dirac flavor on a 2d lattice *(more on this later!)*
- Scalars appear as link variables

Symanzik action and renormalization

Is the continuum limit of this lattice action spoiled by renormalization??

(i) Construct the Symanzik action: shift the vevs of the bosons by $(1/\sqrt{2}\,a)\,\mathbf{1}_k$

(ii) Expand the action for smooth superfields in powers of *a*.

(iii) Include all operators allowed by the exact symmetries (SUSY, lattice reflection, discrete translation) with coefficients known at tree level

(iv) Consider loop corrections to coefficients

(v) Watch out for: relevant operators consistent with lattice symmetry, but not target theory symmetry.

Symanzik action: (*O*=operator, C_O = coefficient)

mass dimension: $\underline{0 = (-2) + (1/2) + (-2) + (7/2 - p) + p}$

$$S = \frac{1}{g_2^2} \int d\theta \int d^2x \sum_O C_O \ O$$

$$\text{dim } \int d\theta = \tfrac{1}{2} \text{ because } \int d\theta \sim \partial_\theta \sim Q \sim \sqrt{P}$$

If *O* has dim = *p*, C_O must have dimension (7/2-*p*)

Radiative corrections at l-loops

$$\delta S = \frac{1}{g_2^2}\int d^2x \sum_{O} \delta C_O\, O$$

An l-loop contribution will be proportional to $(g_2^2)^l$

$$\delta C_O = c_\ell\, a^{p-7/2}(g_2^2 a^2)^\ell$$

(i) c_l is dimensionless, can only depend on *a* logarithmically

(ii) radiative correction is "bad" if *O* is "bad" ***and*** C_O doesn't vanish when *a* vanishes

(iii) So: potential problem only for $p \leq (7/2-2l)$

* *p = mass dimension of O*

Problem if $p \leq (7/2-2l)$ and *O* violates symmetries of the target theory

- $l=0$: No problem, tree level theory is good
- $l=1$: Only a potential problem for $p \leq 3/2$
- $l \geq 2$: never a problem (*p* can't be negative)

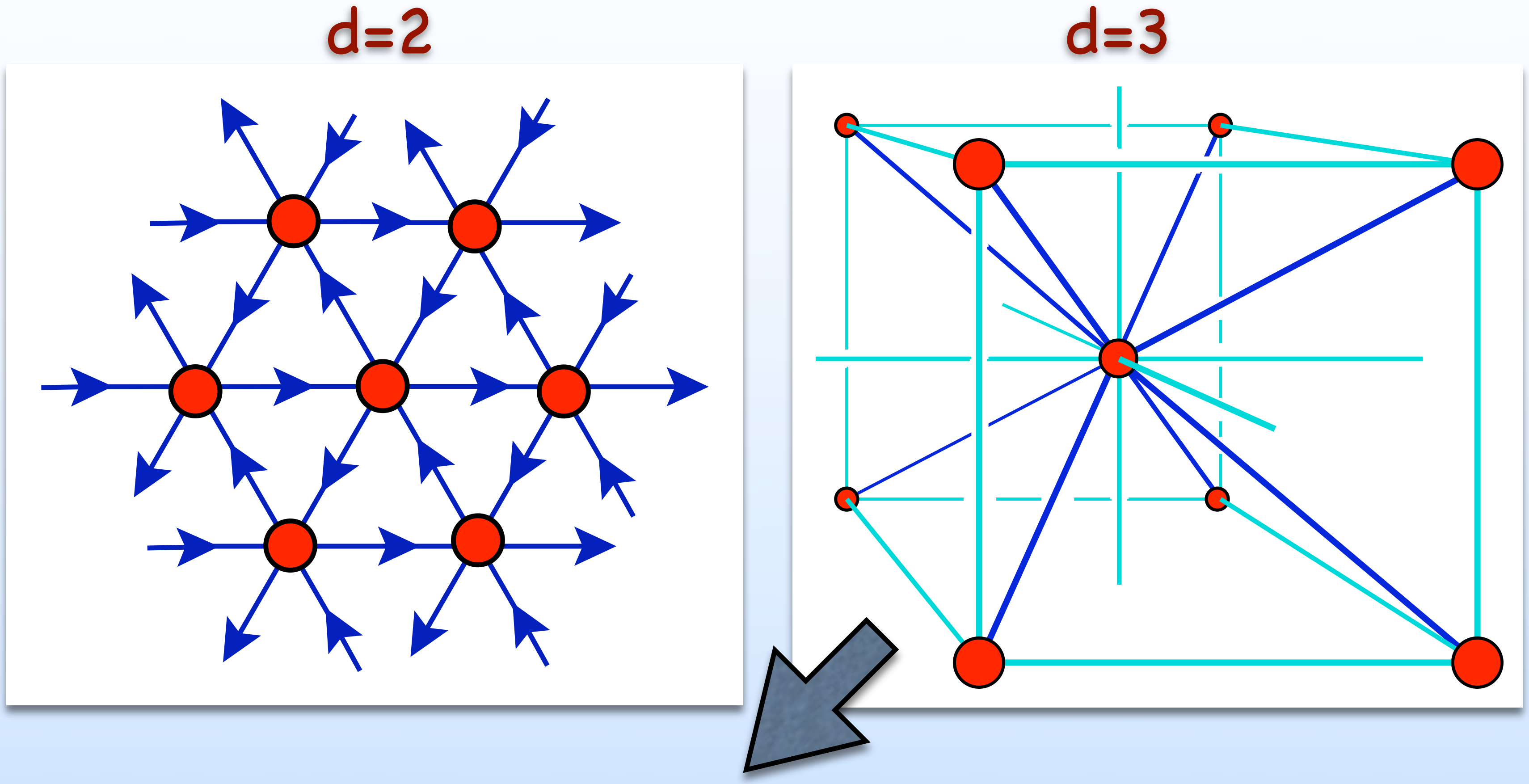
Conclusion: only a problem if we can construct a bad operator *O* with dimension $p \leq 3/2$, consistent with the symmetries of the lattice.

One finds there is no such operator, so no fine tuning is necessary in this theory!

Other SUSY lattices

- For target theory with Q supercharges in d dimensions, lattice has $Q/2^d$ exact SUSY charges. Want $Q/2^d > 0$.
- Can construct lattices for pure Super Yang-Mills target theories with
  - 4 supercharges in d=1,2
  - 8 supercharges in d=1,2,3
  - 16 supercharges in d=1,2,3,4
- Can construct lattices for SYM with 4 supercharges & matter fields in d=2.

Example: target = SYM with 16 supercharges



Site: 2 fermions, 2 real bosons

Light blue link: 2 fermions

Dark blue link: 2 real bosons, 2 fermions

Kaplan and Unsal, hep-lat/0503039

These lattices are pretty, but:

- Most probably have a fermion sign problem
- Only a limited number of theories can be constructed this way (eg, we cannot use these methods to construct a lattice for SUSY QCD with N_f flavors of quarks in d=4 dimensions)
- Analysis of fine-tuning problem not powerful enough to address renormalization of marginal operators in the N=4 SUSY theory in d=4 dimensions
- The orbifold technique: only good for gauge theories?

This technique does not seem like the last word.

Next lecture: return to the fermions for hints on how to extend these lattice constructions.

Part VII

Fermions on the SUSY lattice

- Open questions
- Reduced staggered fermions
- Dirac-Kahler fermions
- “Twisted” supersymmetry

We have learned interesting things about SUSY lattices:

- How SUSY can be realized in terms of component fields on the lattice
- How chiral R-symmetries can emerge in the continuum
- How scalars can appear nontrivially on the lattice
- How there are limitations on what sort of SUSY lattices can be constructed.

But we would like to know more:

- What is the connection to the "Twisted lattice SUSY" approach by Catterall?

- Can we broaden the class of SUSY lattices? (E.g., fewer supercharges, more matter fields)

- Can we use chiral fermion formulations to allow for fewer fermions in non-adjoint representations (eg, for SUSY QCD in d=4)

- Might the renormalization properties be better than expected?

I don’t have the answers.

But it seems that progress might be made by understanding better the connection between lattice SUSY and staggered fermions.

Consider staggered fermions in d=2, as conceived of by Susskind

Dirac action: $\bar{\psi}\gamma_\mu\partial_\mu\psi$

ψ is a 2-component Dirac fermion

Naive discretization:

$$S= \frac{1}{2a}\sum_\mu \bar{\psi}(\mathbf{n})\gamma_\mu\left(\psi(\mathbf{n}+\hat{\mu})-\psi(\mathbf{n}-\hat{\mu})\right)$$

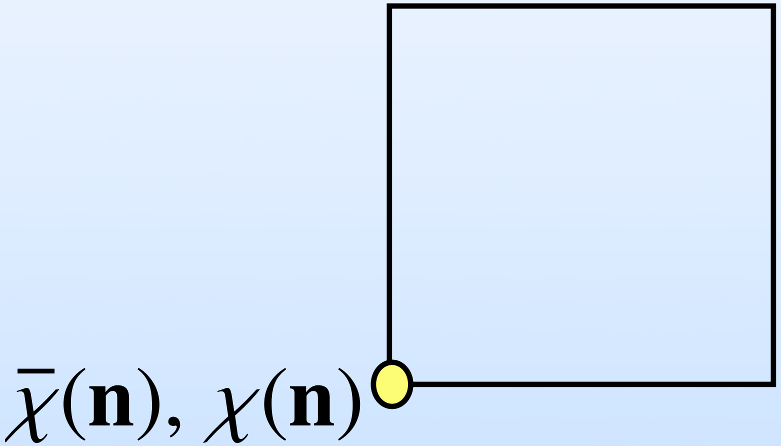
Define: $\psi(\mathbf{n}) = \gamma_2^{n_2}\gamma_1^{n_1}\chi(\mathbf{n})$

$$\bar{\psi}(\mathbf{n}) = \bar{\chi}(\mathbf{n})\gamma_1^{n_1}\gamma_2^{n_2}$$

Action becomes:

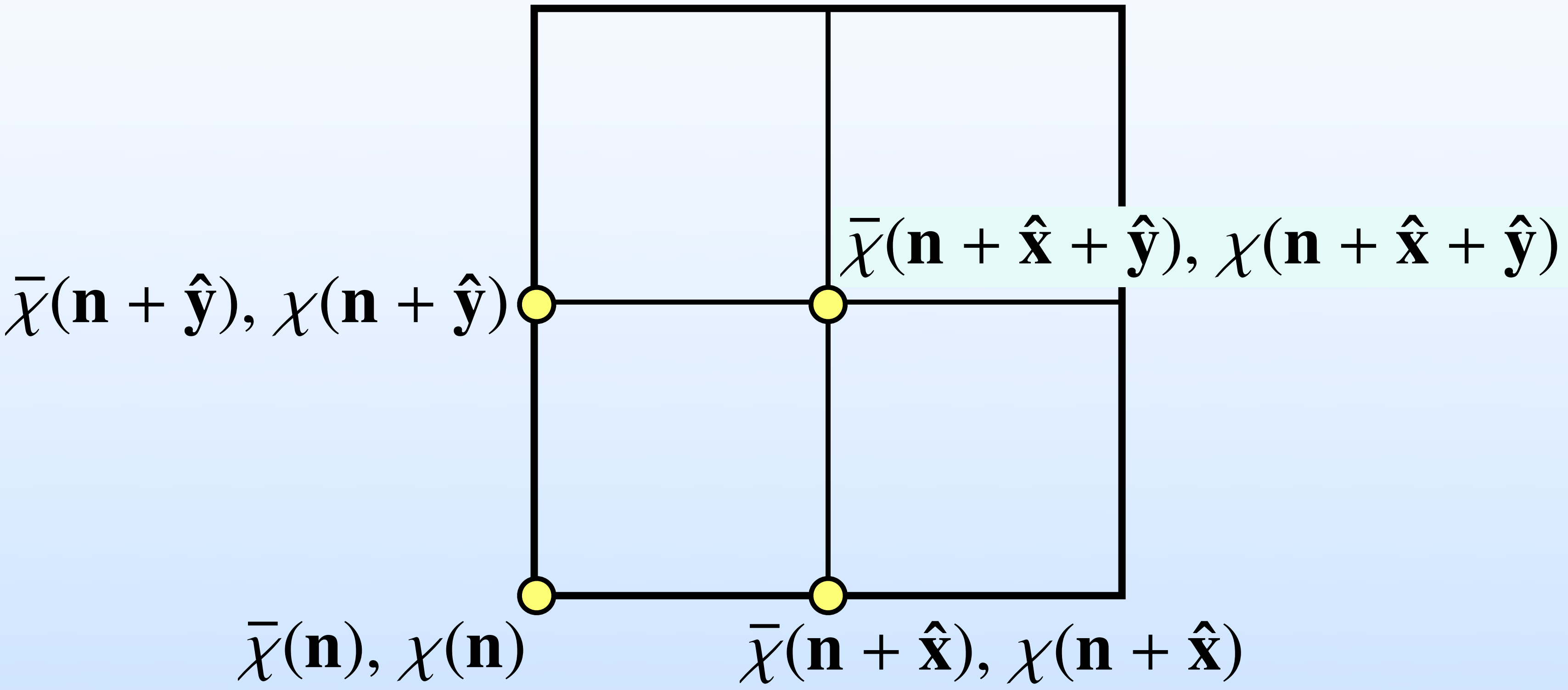
$$S = \sum_{\mathbf{n}} \bar{\chi}(\mathbf{n}) \left[(-1)^{n_2} \left(\chi(\mathbf{n}+\hat{\mathbf{x}}) - \chi(\mathbf{n}-\hat{\mathbf{x}}) \right) + \left(\chi(\mathbf{n}+\hat{\mathbf{y}}) - \chi(\mathbf{n}-\hat{\mathbf{y}}) \right) \right.$$

Note: no more γ structure left. Can make χ into a one-component fermion.



Naive action had 4 Dirac fermions in the continuum; this will have 2 Dirac fermions (4x each 1-component fermion $\bar{\chi}, \chi$)

To make the doubling explicit, go to a lattice twice as coarse:



Now there is no doubling, and you see all 8 degrees of freedom explicitly

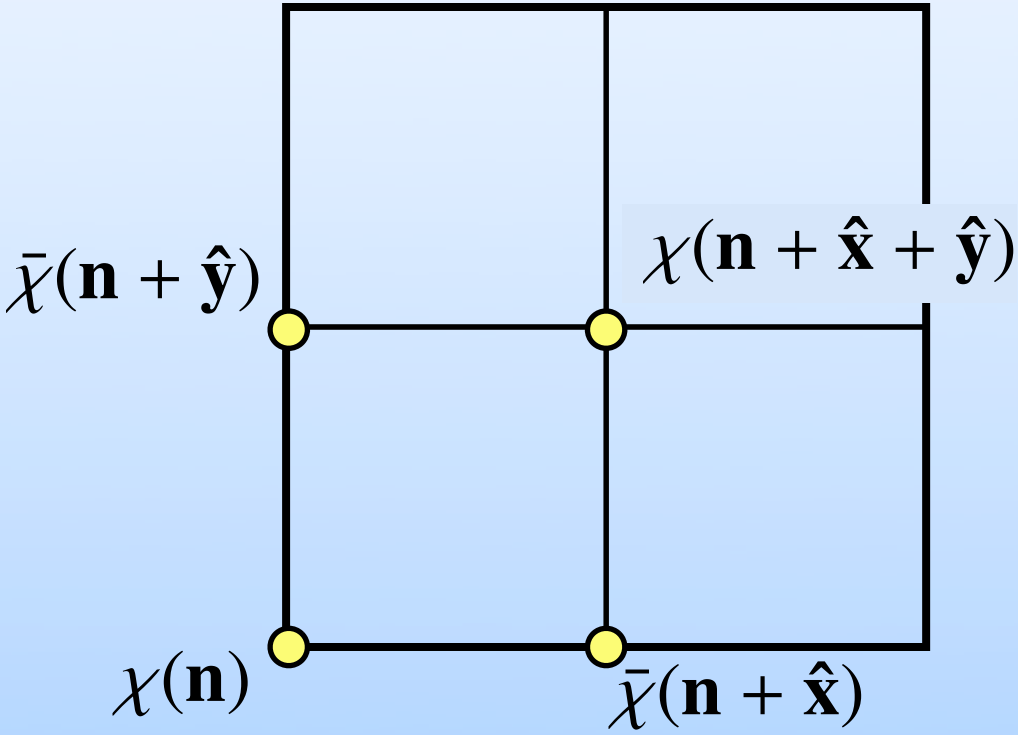
Next: note that in action $\bar{\chi}$ on odd sites only interacts with χ on even sites, & converse. So only keep odd site $\bar{\chi}$ and even site χ.

Eliminate:

$\bar{\chi}(\mathbf{n}), \bar{\chi}(\mathbf{n}+\hat{\mathbf{x}}+\hat{\mathbf{y}})$

$\chi(\mathbf{n}+\hat{\mathbf{x}}), \chi(\mathbf{n}+\hat{\mathbf{y}})$

Now only get one 2-component Dirac fermion in the continuum...just like our SUSY lattice



To make identification obvious -

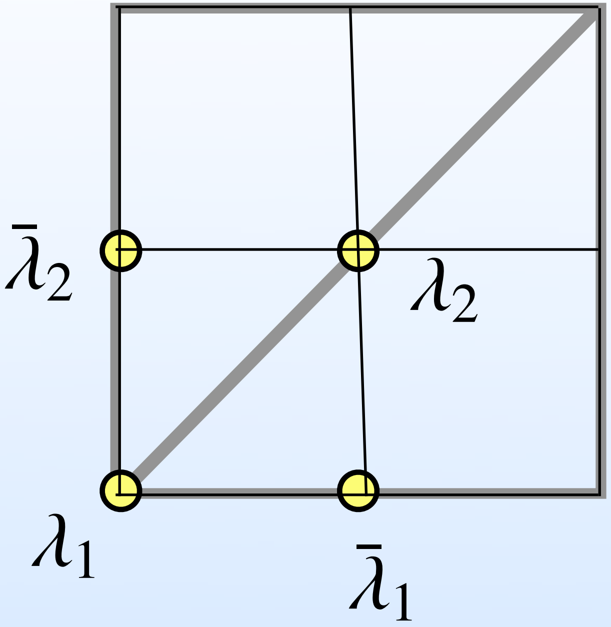
Rename:

$$\chi(\mathbf{n}) \rightarrow \lambda_1(\mathbf{n})$$

$$\chi(\mathbf{n}+\hat{\mathbf{x}}+\hat{\mathbf{y}}) \rightarrow \lambda_2(\mathbf{n})$$

$$\bar{\chi}(\mathbf{n}+\hat{\mathbf{x}}) \rightarrow \bar{\lambda}_1(\mathbf{n})$$

$$\bar{\chi}(\mathbf{n}+\hat{\mathbf{y}}) \rightarrow \bar{\lambda}_2(\mathbf{n})$$



In terms of these variables, the Susskind action

$$S = \sum_{\mathbf{n}} \bar{\chi}(\mathbf{n})\left[(-1)^{n_2}\left(\chi(\mathbf{n}+\hat{\mathbf{x}})-\chi(\mathbf{n}-\hat{\mathbf{x}})\right)+\left(\chi(\mathbf{n}+\hat{\mathbf{y}})-\chi(\mathbf{n}-\hat{\mathbf{y}})\right)\right.$$

becomes equivalent to our free fermion lattice action (up to unimportant overall sign)

The connection between staggered fermions and geometry

d=2 dimensions: the Euclidian “Lorentz” group = SO(2).

The flavor symmetry of a single Dirac fermion is also U(1)=SO(2).

Somehow the reduced staggered fermion scrambles up this SO(2)xSO(2) symmetry, but the derivation makes it hard to see.

Much clearer in the equivalent Dirac-Kahler formulation (Kahler 1962; Rabin 1981; Becher & Joos 1982)

Disclaimer: For the rest of this section, be wary of signs & numerical factors!

A quick summary of p-forms, useful for describing totally anti-symmetric tensors in a geometric context:

$$F = f + f_\mu \, \mathrm{d}x_\mu + \frac{1}{2!} f_{[\mu\nu]} \, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu + \dots$$

0-form 1-form 2-form +...

All f's are functions of x. Two types of differential operators:

$$\mathrm{d}\, F = \partial_\mu f \, \mathrm{d}x_\mu + \frac{1}{2!} \partial_\mu f_\nu \, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu + \dots$$

curl: p -> p+1

$$\delta\, F = \partial_\mu f_\mu + \partial_\mu f_{[\mu\nu]} \mathrm{d}x_\nu + \dots$$

div: p -> p-1

The Dirac equation can be formulated in terms of p-forms, for the right number of flavors (Kahler)

Example: d=2, 2 flavors of Dirac fermion. Write as a 2x2 matrix, and then expand in the gamma matrix basis:

$$\Psi_{\alpha i} = \left[\psi + \psi_\mu\gamma_\mu + \tfrac{1}{2}\psi_{[12]}(\gamma_1\gamma_2 - \gamma_2\gamma_1)\right]_{\alpha i}$$

☞ Under SO(2)$_L$ x U(2)$_f$ symmetry, the fermion transforms as $\Psi \to \Lambda\Psi U^\dagger$

☞ The components $\psi,\ \psi_\mu,\ \psi_{[\mu\nu]}$ transform as **tensors** under the diagonal subgroup

$$SO(2) \subset SO(2)_L \times U(2)_f$$

Now that the fermions are classified as tensors, instead of spinors, they have a natural geometric interpretation when latticizing them:

ψ 0-forms → sites

ψ_μ 1-forms → links

$\psi_{[\mu\nu]}$ 2-forms → plaquettes

Furthermore, the d and δ operations have natural interpretations as lattice difference operators.

Latticized Dirac-Kahler fermions are equivalent to staggered fermions.

One can produce *reduced* staggered fermions from Dirac-Kahler for real representations (like adjoints)

This formulation makes it clear that staggered fermions have a well defined geometric significance, and that the point group of the lattice lies in a nontrivial subgroup of (Lorentz x Flavor), as we have seen in our SUSY lattices.

In supersymmetry, the supercharges are spinors. Like fermions, they do not have any natural geometric interpretation.

Kahler trick: Classify supercharges as antisymmetric tensors under Lorentz x R-symmetry

$$\left\{Q,\ Q_\mu,\ Q_{[\mu\nu]},\ldots\right\}$$

For a large number of supercharges, there could be multiple copies.

Called "twisted supersymmetry"

Now both fermions and bosons can be given geometric meaning and assigned roles on a lattice:

0 index tensors → sites

1 index tensors → links

2 index tensors → plaquettes

Only the 0-index supercharges (located at sites) are unbroken by the latticization, since only they interchange bosons and fermions at the same place on the lattice.

Using "twisted SUSY" to construct SUSY lattices has been pioneered by Simon Catterall.

The resulting lattices for SYM have been shown by Unsal to be equivalent to those produced via orbifolding.

This approach naturally leads to staggered fermions. Is there either a generalization or alternative that leads to overlap/domain wall fermions? Not known.

Part VIII

Some idle thoughts about lattice supergravity

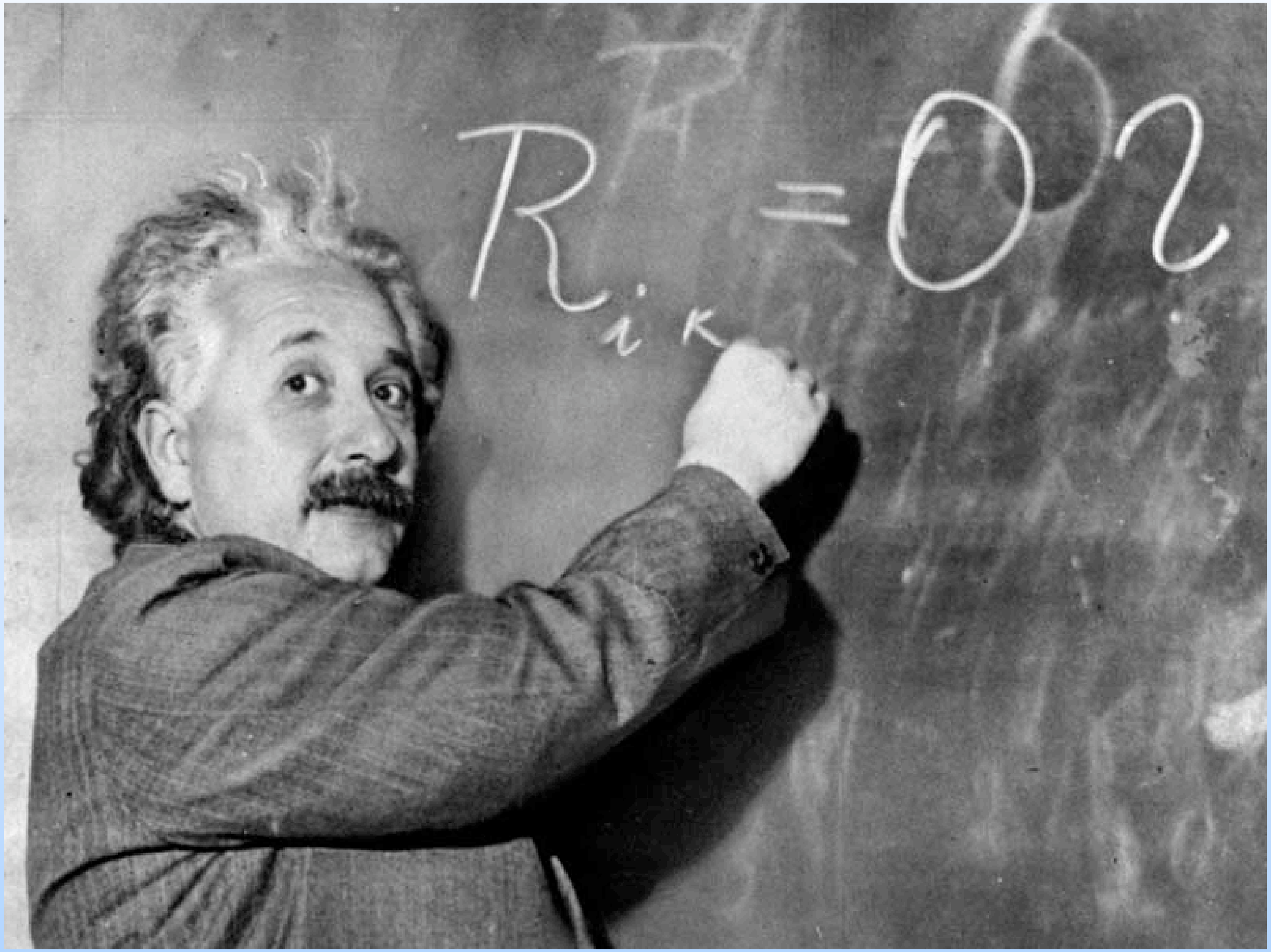
- Staggered gravitinos?
- A lattice for vierbeins?
- Where in the world does this lattice live??

Can we use our technology to construct lattice supergravity?

The wishful thinking:

- Lattice gravity is very confusing!
- We have machinery for creating supersymmetric lattices; link length is dynamical
- Perhaps without thinking much we can construct a supersymmetric lattice with a spin 3/2 fermion?
- Supersymmetric partner will then be a graviton
- Gravity without tears!?

Then:



Now:

Step 1: Can we construct staggered Rarita-Schwinger (spin 3/2) fermions?

4d: $\psi_{m\alpha}$

4-vector index

Dirac spinor index

Continuum action: $\epsilon_{mnpq}\,\psi_m^T C\gamma_n\gamma_5\partial_p\psi_q$

Gauge freedom: $\psi_m \rightarrow \psi_m + \partial_m\chi$

$\chi =$ arbitrary Majorana spinor

Can naively discretize:

$$\epsilon_{mnpq}\,\psi_m^T C\gamma_n\gamma_5\partial_p\psi_q$$

$$\rightarrow \frac{1}{2a}\,\epsilon_{mnpq}\,\psi_m^T(\mathbf{n})C\gamma_n\gamma_5\left(\psi_q(\mathbf{n}+\hat{\mathbf{e}}_p)-\psi_q(\mathbf{n}-\hat{\mathbf{e}}_p)\right)$$

Exact fermionic gauge symmetry is preserved.

Next, perform the analog of Susskind spin diagonalization:

$$\psi_m(\mathbf{n}) = \gamma_m\left(\gamma_1^{n_1}\cdots\gamma_4^{n_4}\right)\lambda_m(\mathbf{n})$$

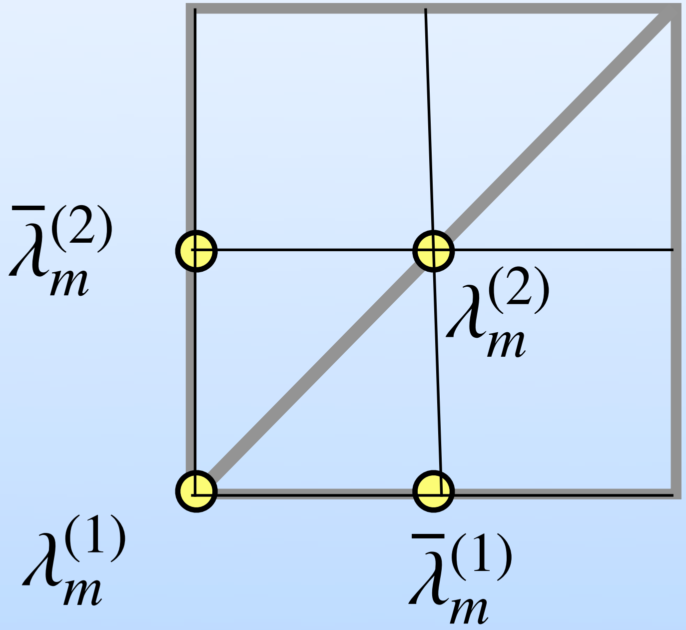
Allows one to reduce number of modes to 1/4

Looks like staggered fermion, but with vector index, different phases.

With reduced staggered gravitinos, left with a minimum of 2 gravitinos on the lattice...appropriate for N=2 SUGRA

M. Endres, D.K.

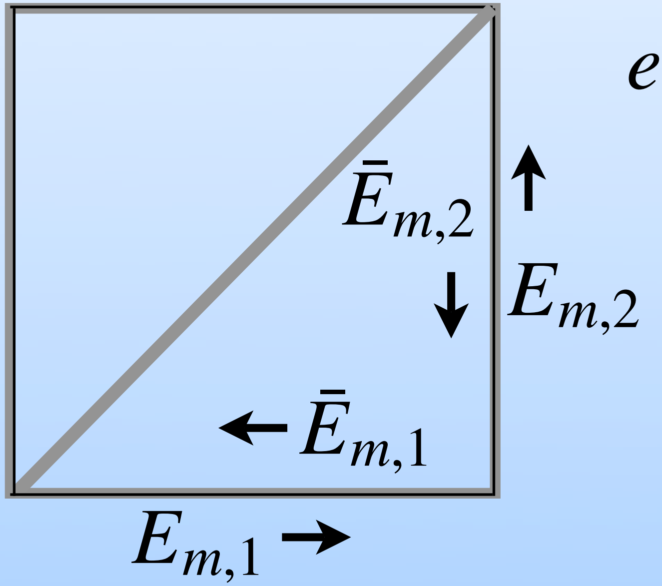
As a first step, we considered instead 4 supercharge SUGRA in 2d. Get a familiar lattice:



In SUGRA, partner of the gravitino is the vierbein (square root of the metric) e_{am}

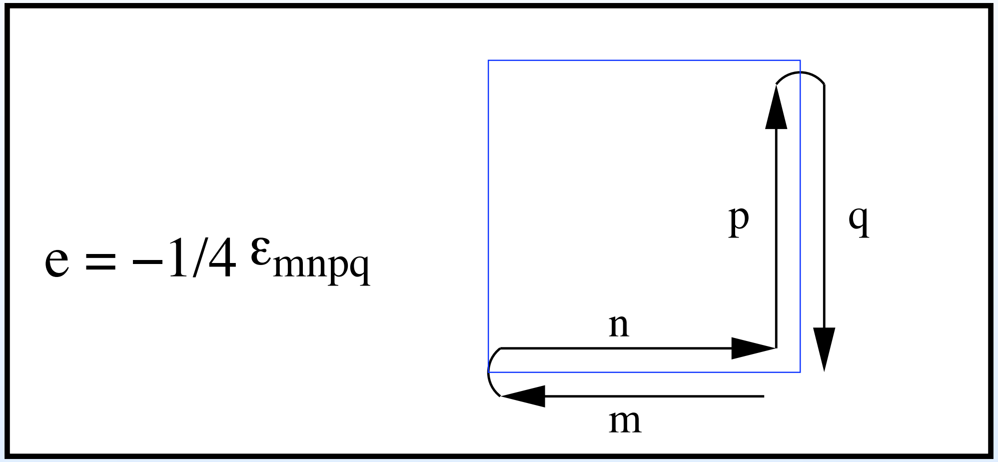
$$e_{am} e^a_n = g_{mn} \ , \qquad e_{am} e^m_b = \eta_{ab}$$

"a" is a flat space index, and knows about Lorentz SO(4); "m" is a curved space index and does not. Using SO(4), can assign the vierbein to the lattice as well:



$$e_{m,\alpha\dot{\beta}} \equiv e^a_m \sigma_{a,\alpha\dot{\alpha}} \equiv \begin{pmatrix} E_{m,1} & E_{m,2} \\ -\bar{E}_{m,2} & \bar{E}_{m,1} \end{pmatrix}$$

SUGRA also needs things like $e=\det(e_{ma})$, $e^{-1}=e^{m}{}_{a}$; those also have simple forms on the lattice. For example:



$$e = -1/4\, \varepsilon_{mnpq}$$

The rest of the SUGRA multiplet can be similarly represented on the lattice.

The problem we couldn’t hide from:

Our lattice knew all about flat space indices and the local Lorentz group, but curved space indices had no role.

Couldn’t figure out what space the lattice represented! How to define covariant derivatives!

Attempts to put Supergravity on the lattice has both its encouraging and discouraging features.

Perhaps it's stupid, or perhaps it is just waiting for the next good idea...

Conclusions:

- Supersymmetry is a fascinating symmetry, and for at least a few theories we know now how to define a nonperturbative lattice regulator.

- Perhaps some of these lattices will some day be numerically tractable; perhaps they will some day be of use for better understanding field theoretical descriptions of quantum gravity

- Is a supergravity lattice theory possible to construct? Maybe: currently a mix of encouraging and discouraging results.

- More structure here to be discovered? It would seem so.

Many thanks to my collaborators on lattice supersymmetry (in alphabetical order):

- A. Cohen
- M. Endres
- E. Katz
- M. Unsal