Anomalies, edge states, and topology

Three lectures delivered at the EuroPLEx Summer School 2022, Benasque, June 6-8, 2022

David B. Kaplan

Institute for Nuclear Theory, University of Washington, Seattle WA 98195

E-mail: dbkaplan@uw.edu

Contents

1 Introduction

Anomalies are a fascinating phenomenon in quantum field theory: obstructions to defining quantum versions of classical symmetries of the action, or even to defining the path integral for what looks like a sensible action. They are intimately tied to topology and the massless states that appear at the boundaries between quantum matter in different topological phases. These edge states play a role in condensed matter systems – topological insulators and superconductors, with the Integer Quantum Hall Effect being a particular manifestation. They also play a role in simulating quantum field theories on the lattice with chiral symmetry; they may provide the key to a nonperturbative formulation of chiral gauge theories (such as the Standard Model), as well as proposals for creating qubits for quantum computation which are topologically protected from certain types of errors. This subject of anomalies has been developed over the past 53 years, and yet remains very much at the forefront of contemporary physics. As a result, the subject is vast, and these three lectures will just touch on a minuscule subset of topics that I find interesting, and which hopefully tell a coherent story.

2 The chiral anomaly in 1+1 dimensions.

2.1 Fermions in 1+1 dimensions

I will start by discussing the chiral anomaly in 1+1 dimensions from several different viewpoints. The calculations are simpler than for anomalies in higher dimensions, but are similar. Consider a free, massive Dirac fermion in $d = 1 + 1$ dimensions, whose coordinates we will call $x^0 = t$, $x^1 = x$. The Lagrange density is

$$\mathcal{L} = \bar{\psi}\left(i\partial_\mu\gamma^\mu - m\right)\psi \ , \tag{2.1}$$

where ψ is a 2-component spinor, and the γ-matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \ . \tag{2.2}$$

A convenient representation for the γ-matrices in terms of the Pauli matrices is

$$\gamma^0 = \sigma_1 \ , \qquad \gamma^1 = -i\sigma_2 \ , \qquad \sigma^{01} = \frac{i}{4}\left[\gamma^0, \gamma^1\right] = \frac{i}{2}\sigma_3 \tag{2.3}$$

and Lorentz transformation of ψ takes the form

$$\psi(x) \to e^{\theta\sigma_3/2}\psi(\Lambda^{-1}x) \ , \qquad \Lambda_\mu^{\ \nu} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \ . \tag{2.4}$$

Note that the Lorentz boost matrix $e^{\theta\sigma_3/2}$ is diagonal and real. That means that Dirac fermions are not the irreducible representation of the Lorentz group. Instead, the irreducible representation is a 1-component real fermion, where in our basis the single component resides in either the upper or lower component of the 2-component spinor. These are called Majorana-Weyl fermions, and are eigenstates of the matrix γ_χ (the analog of γ_χ in 3+1 dimensions)[1], defined by

$$\gamma_\chi = \gamma^0\gamma^1 = \sigma_3 \ , \qquad \{\gamma_\chi, \gamma^\mu\} = 0 \ , \qquad [\gamma_\chi, \sigma^{\mu\nu}] = 0 \ . \tag{2.5}$$

In our basis, $\gamma_\chi = \sigma_3$ with eigenvalues ± 1. A field with eigenvalue 1 is called "right-handed", and with eigenvalue -1 is called "left-handed". A Majorana-Weyl fermion is described by a real field which is either RH or LH, which can be represented in our basis as a 2-component spinor with a zero in either the lower or upper component, and a real function of spacetime in the other component.

From Majorana-Weyl spinors we can create various useful reducible representations. From two LH or two RH Majorana-Weyl spinors we can construct a complex field which is an eigenstate of chirality, which we call RH and LH Weyl spinors respectively. From one LH and one RH Majorana-Weyl spinor we can construct a real spinor where both components are nonzero, called a Majorana spinor. Finally, from two LH and two RH Majorana-Weyl spinors (or equivalently a LH plus a RH Weyl spinor, or two Majorana spinors) we make a complex 2-component spinor which is the Dirac spinor.

Consider first a Dirac field ψ. To avoid basis dependence, we can define projection operators

$$P_\pm = \frac{1 \pm \gamma_\chi}{2} \ , \qquad P_\pm^2 = P_\pm \ , \qquad P_+ + P_- = 1 \ . \tag{2.6}$$

(where "1" means the 2×2 unit matrix). Then we define

$$\psi_R = P_+\psi \ , \quad \psi_L = P_-\psi \ , \quad \bar\psi_L = \psi_L^\dagger\gamma^0 = \bar\psi P_+ \ , \quad \bar\psi_R = \psi_R^\dagger\gamma^0 = \bar\psi P_- \ . \tag{2.7}$$

With the above definitions, writing $\psi = \psi_L + \psi_R$ and plugging back into our Lagrange density in eq. (2.1) we find we can rewrite it as

$$\mathcal{L} = \bar\psi_L i\partial\!\!\!/ \psi_L + \bar\psi_R i\partial\!\!\!/ \psi_R - \left(m\bar\psi_L\psi_R + \text{h.c.}\right) \tag{2.8}$$

Since both $\psi_{R,L}$ are complex, we can define independent $U(1)$ phase redefinitions of the fields, or equivalently a $U(1)_V$ transformation ("V" = "vector"), where we rotate

[1]Since I will be discussing chirality in both 1+1 and 3+1 dimensions I will use the notation γ_χ for both, rather than γ_3 and γ_χ.

them the same way, and $U(1)_A$ transformation ("A"="axial") where we rotate them oppositely:

$$\begin{aligned} U(1)_V: &\qquad \psi \to e^{i\alpha}\psi\ , &\qquad \bar\psi \to \bar\psi e^{-i\alpha} \\ U(1)_A: &\qquad \psi \to e^{i\gamma_\chi\beta}\psi\ , &\qquad \bar\psi \to \bar\psi e^{i\beta\gamma_\chi}\ . \end{aligned} \tag{2.9}$$

and the associated currents are

$$j_V^\mu = \bar\psi\gamma^\mu\psi\ , \qquad j_A^\mu = \bar\psi\gamma^\mu\gamma_\chi\psi\ . \tag{2.10}$$

The $U(1)_V$ transformation is an exact symmetry of the Lagrangian eq. (2.8), while the fermion mass explicitly breaks the $U(1)_A$ symmetry since it couples ψ_L and ψ_R to each other. Noether's theorem tells us that

$$\partial_\mu j^\mu = 0, \qquad \partial_\mu j_A^\mu = 2im\bar\psi\gamma_\chi\psi\ . \tag{2.11}$$

If we want to consider a theory of just a RH Weyl fermion, then we erase ψ_L from eq. (2.8), and we see that necessarily describes a massless particle. It still has a classical $U(1)$ symmetry, with the current $j_R^\mu = \bar\psi_R\gamma^\mu\psi_R$. If instead we want to describe a Majorana fermion, we require that both ψ_L and ψ_R be real. In this case a mass term is still allowed, but there is no $U(1)$ symmetry, although there is still a $\psi \to -\psi$ symmetry. Finally, a theory of a Majorana-Weyl fermion can have neither a mass term, nor a conserved fermion number[2].

2.2 Heuristic explanation of the chiral anomaly

Returning to the Dirac fermion, if we set $m = 0$ it appears that both the $U(1)_V$ and $U(1)_A$ currents are conserved and can be coupled to independent $U(1)$ gauge fields. To do so we replace the ordinary derivative ∂_μ in the Dirac action by the covariant derivative $D_\mu = \partial_\mu - ieA_\mu - igB_\mu\gamma_\chi$. However, one finds that in the quantum theory, if we gauge the $U(1)_V$ symmetry, then $U(1)_A$ is no longer a symmetry even in the massless limit, $\partial_\mu j_A^\mu \neq 0$, and we cannot gauge it. The converse is also true. This is called a "chiral anomaly". Anomalies turn out to be very relevant both for phenomenology, and central for understanding the challenges for implementing chiral symmetry in lattice field theory. The reason anomalies affect chiral symmetries is that regularization requires a cut-off on the infinite number of modes above some mass scale, while chiral symmetry is incompatible with fermion masses[3].

[2]The situation in 2+1 and 3+1 dimensions is different. There are no Majorana-Weyl representations in either case; while in $2+1$ dimensions there are no Weyl fermions either. In $3 = 1$ dimensions, a single Weyl fermion can be given a Lorentz-invariant mass which breaks fermion number to Z_2, and is equivalent to a massive Majorana fermion. In fact, one does not encounter Majorana-Weyl fermions again until one reaches $9 + 1$ dimensions.

[3]Dimensional regularization is not a loophole, since chiral symmetry cannot be analytically continued away from odd space dimensions.

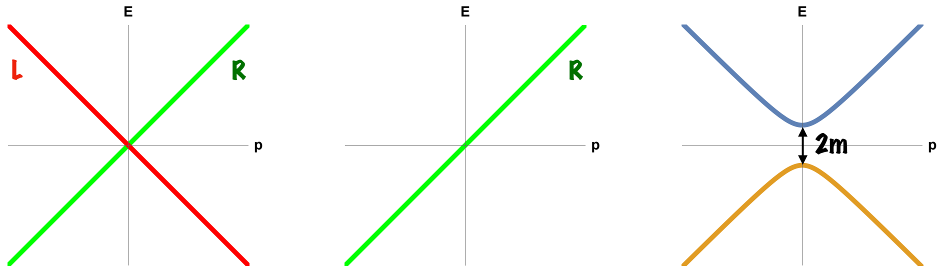

Figure 1. *The spectrum of (i) a free massless Dirac fermion in* $d = 1+1$*, (ii) a free massless RH chiral fermion, (iii) a free massive Dirac fermion. In the first case both LH and RH fermion numbers are independently conserved — or equivalently, both fermion number and axial charge are conserved; in the second case there is a conserved RH fermion number; for the massive case there is only a conserved fermion number.*

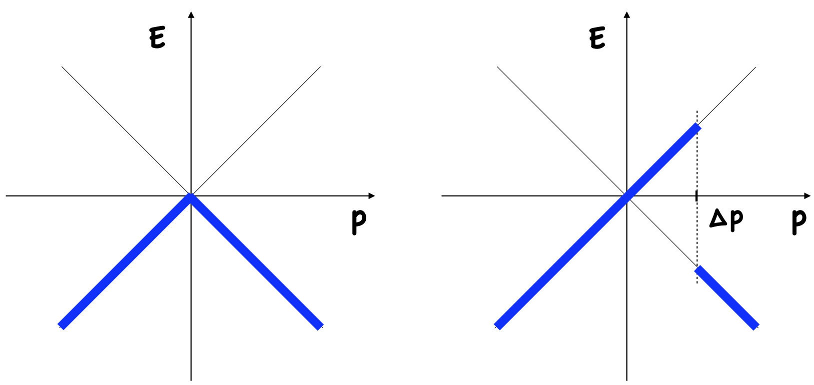

Figure 2. *On the left: the ground state for a theory of a single massless Dirac fermion in* $(1+1)$ *dimensions; on the right: the theory after application of an adiabatic electric field with all states shifted to the right by* Δp*, given in eq.* (2.12)*. Filled states are indicated by the heavier blue lines.*

A simple way to derive anomalies (and in some ways, overly simple) is to look at what happens to the ground state of a theory with a single flavor of massless Dirac fermion in $(1+1)$ dimensions with charge q in the presence of an electric field. Suppose one adiabatically turns on a constant positive electric field $E(t)$, then later turns it off; the equation of motion for the fermion is $\frac{dp}{dt} = qE(t)$ and the total change in momentum is

$$\Delta p = q \int E(t)\, dt \ . \tag{2.12}$$

Thus the momenta of both left- and right-moving modes increase; if one starts in the ground state of the theory with filled Dirac sea, after the electric field has turned off, both the right-moving and left-moving sea levels have shifted to the right as in Fig. 2. The the final state differs from the original by the creation of particle- antiparticle pairs: right moving particles and left moving antiparticles. Thus while there is a fermion current in the final state, fermion number has not changed. This is what one would expect from conservation of the $U(1)$ current:

$$\partial_\mu j^\mu = 0 \ , \tag{2.13}$$

However, recall that right-moving and left-moving particles have positive and negative chirality respectively; therefore the final state in Fig. 2 has net axial charge, even though the initial state did not. This is peculiar, since the coupling of the electromagnetic field in the Lagrangian does not violate chirality. We can quantify the effect: if we place the system in a box of size L with periodic boundary conditions, momenta are quantized as $p_n = 2\pi n/L$. The change in axial charge is then

$$\Delta Q_A = 2\frac{\Delta p}{2\pi/L} = \frac{q}{\pi}\int d^2x\, E(t) = \frac{q}{2\pi}\int d^2x\, \epsilon_{\mu\nu}F^{\mu\nu} \ , \tag{2.14}$$

where I expressed the electric field in terms of the field strength F, where $F^{01} = -F^{10} = E$. This can be converted into the local equation using $\Delta Q_A = \int d^2x\, \partial_\mu j^\mu_A$, a modification of eq. (2.11) (with $m = 0$):

$$\partial_\mu j^\mu_A = \frac{q}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ . \tag{2.15}$$

The term on the right is the axial anomaly in $1+1$ dimensions.

So how did an electric field end up violating chiral charge? Note that this analysis relied on the Dirac sea being infinitely deep. If there had been a finite number of negative energy states, then they would have shifted to higher momentum, but there would have been no change in the axial charge. With an infinite number of degrees of freedom, though, one can have a "Hilbert Hotel": the infinite hotel which can always accommodate another visitor, even when full, by moving each guest to the next room and thereby opening up a room for the newcomer. This should tell you that it will not be straightforward to represent chiral anomalies on the lattice: a lattice field theory approximates quantum field theory with a finite number of degrees of freedom — the lattice may be a big hotel, but it is quite conventional. In such a hotel there can be no anomaly, since there is no ambiguity about how many occupants it has.

This method of deriving the anomaly gives the correct answer, but is a bit too simplistic. For one thing, there should be no need to assume that the gauge field must change adiabatically. For another, it doesn't help one figure out what happens in the case where there is a fermion mass and a gap, where the correct answer is that one just adds together the anomalous and classical symmetry violation, modifying eq. (2.11) to read

$$\partial_\mu \overline{\psi}\gamma^\mu\gamma_\chi\psi = 2im\,\overline{\psi}\gamma_\chi\psi + \frac{q}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ . \tag{2.16}$$

2.3 The anomaly from a Feynman diagram

Let's see how to derive eq. (2.16) from a Feynman diagram. We need to regulate the theory and I will do so by including a Pauli-Villars regulator. To understand the effect of a Pauli-Villars regulator, consider the Euclidean Dirac operator $\not{D}$. The unregulated fermion determinant in Euclidean spacetime can be formally written as

$$\det(\not{D} + m) = \prod_n (i\lambda_n + m) \ , \tag{2.17}$$

where $i\lambda_n$ are the eigenvalues of the antihermitian $\not{D}$ operator. With a Pauli-Villars regulator, this is replaced by

$$\lim_{\Lambda\to\infty} \frac{\det(\not{D} + m)}{\det(\not{D} + \Lambda)} = \lim_{\Lambda\to\infty} \prod_n \frac{i\lambda_n + m}{i\lambda_n + \Lambda} \ . \tag{2.18}$$

We see that for fixed $\Lambda \gg m > 0$, the effect of eigenvalues $|\lambda_n| \gg \Lambda$ are damped out – regulated – and then at the end of the calculation, the regulator mass Λ is taken to infinity, so that the contribution to low lying eigenvalues to the partition function is maintained. Operationally, this is like adding a field with Dirac action corresponding to mass Λ, but with Bose statistics in order to get the inverse determinant. So when computing Feynman diagrams, we add a fermion with mass Λ, but do not include the factor of (-1) for every closed loop.

First, compute the pseudoscalar density, given by the diagram in Fig. 3 with an insertion of $\bar{\psi}\gamma_\chi\psi$:

$$\begin{aligned}\langle\bar{\psi}\gamma_\chi\psi\rangle &= \lim_{\Lambda\to\infty}(-1)(iq)\int\frac{d^2\ell}{(2\pi)^2}\,\mathrm{Tr}\,\gamma_\chi\frac{i}{\not{\ell}+\not{p}-m-i\epsilon}\gamma^\nu\frac{i}{\not{\ell}-m-i\epsilon}\tilde{A}_\nu(p) - (m\leftrightarrow\Lambda)\\ &= \lim_{\Lambda\to\infty}\frac{q}{2\pi}p_\mu\tilde{A}_\nu(p)\int_0^1 dx\frac{m\epsilon^{\mu\nu}}{(m^2-p^2x(1-x)+i\epsilon)} - (m\leftrightarrow\Lambda) \ ,\end{aligned} \tag{2.19}$$

where $\tilde{A}_\nu(p)$ is the Fourier transform of $A_\nu(x)$. By looking at the fermion contribution, we see that in the limit $\Lambda \to \infty$, the Pauli-Villars regulator contribution scales as $1/\Lambda$ for large Λ, and so it can be ignored in the $\Lambda \to \infty$ limit.

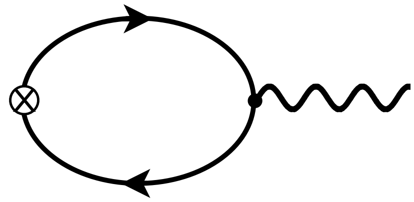

Figure 3. *The anomaly diagrams in 1+1 dimensions involving a fermion of mass m and electric charge q, with an insertion of either $\overline{\psi}\gamma_\chi\psi$ or $\overline{\psi}\gamma^\mu\gamma_\chi\psi$ at the X; equivalent diagrams with a Pauli-Villars field in the loop with mass Λ and charge q are subtracted, and then the limit $\Lambda\to\infty$ is taken.*

For the axial current, given by the diagram in Fig. 3 with an insertion of $\bar\psi\gamma^\mu\gamma_\chi\psi$, one finds

$$\begin{aligned}
\langle\bar\psi\gamma^\mu\gamma_\chi\psi\rangle &= \lim_{\Lambda\to\infty}(-1)(iq)\int\frac{d^2\ell}{(2\pi)^2}\,\mathrm{Tr}\;\gamma_\mu\gamma_\chi\frac{i}{\not\ell+\not p-m-i\epsilon}\gamma_\nu\frac{i}{\not\ell-m-i\epsilon}\tilde A_\nu(p)-(m\leftrightarrow\Lambda)\\
&= \lim_{\Lambda\to\infty}-\frac{q}{2\pi}\tilde A_\nu(p)\int_0^1 dx\frac{m^2\epsilon^{\mu\nu}+x(1-x)\left(p^2\epsilon^{\mu\nu}-2p^\nu p_\alpha\epsilon^{\mu\alpha}\right)}{m^2-p^2x(1-x)+i\epsilon}-(m\leftrightarrow\Lambda)\\
&= -\frac{q}{2\pi}\tilde A_\nu(p)\int_0^1 dx\left(\frac{m^2\epsilon^{\mu\nu}+x(1-x)\left(p^2\epsilon^{\mu\nu}-2p^\nu p_\alpha\epsilon^{\mu\alpha}\right)}{m^2-p^2x(1-x)+i\epsilon}-\epsilon^{\mu\nu}\right)\\
&= -\frac{q}{\pi}\tilde A_\nu(p)\int_0^1 dx\frac{x(1-x)}{m^2-p^2x(1-x)+i\epsilon}\left(p^2\epsilon^{\mu\nu}-p^\nu p_\alpha\epsilon^{\mu\alpha}\right)
\end{aligned}\tag{2.20}$$

Notice that in this case, the effect of the regulator did not vanish as $\Lambda\to\infty$; it is responsible for the $-\epsilon_{\mu\nu}$ term on the right in the third line (obtained by replacing m by Λ in the expression to the left and taking $\Lambda\to\infty$). Since under a gauge transformation $A_\nu(x)\to A_\nu(x)+\partial_\nu\lambda(x)$, it follows that $\tilde A_\nu(p)$ shifts proportionally to p_ν. Therefore it is gratifying that the quantity in the last line vanishes when contracted with p_ν, a sign of gauge invariance, or equivalently, conservation of the vector current.

If we contract the axial current with $-ip_\mu$ and combine results from eq. (2.20) and eq. (2.19) we find

$$\begin{aligned}
-ip_\mu\langle\bar\psi\gamma^\mu\gamma_\chi\psi\rangle-2im\langle\bar\psi\gamma_\chi\psi\rangle &= -i\frac{q}{\pi}\epsilon^{\mu\nu}p_\mu\tilde A_\nu(p)\int_0^1 dx\,\frac{m^2-p^2x(1-x)}{m^2-p^2x(1-x)}\\
&= -i\frac{q}{\pi}\epsilon^{\mu\nu}p_\mu\tilde A_\nu(p)
\end{aligned}\tag{2.21}$$

Taking the Fourier transform gives us

$$\partial_\mu j^\mu_A = 2im\bar\psi\gamma_\chi\psi+\frac{q}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}\ , \tag{2.22}$$

reproducing our result in eq. (2.16).

2.4 Fujikawa's derivation of the anomaly from the path integral measure

A third and complementary way to understand the anomaly is due to Fujikawa, who explained that just because a field transformation might be a symmetry of the action, it is not a symmetry of the path integral unless the transformation of the integration measure gives a trivial Jacobian. In fact, when one performs an axial transformation, in the process of regulating the Jacobian one recovers the axial anomaly.

While we are interested in the anomaly in global (spacetime-independent) chiral transformations, consider for now the spacetime-dependent transformation

$$\psi \to \exp(i\alpha(x)\gamma_\chi)\psi\ , \qquad \bar\psi \to \bar\psi \exp(i\alpha(x)\gamma_\chi)\ , \tag{2.23}$$

where $\alpha(x)$ is an arbitrary function. This is to be thought of as a transformation of integration variables in the path integral. For small α we have

$$\mathcal{L} \to \mathcal{L} - \partial_\mu\alpha\, \bar\psi\gamma^\mu\gamma_\chi\psi + 2i\alpha m\bar\psi\gamma_\chi\psi + O(\alpha^2)\ , \tag{2.24}$$

The expectation value of some operator $\mathcal{O}$ is defined as

$$\langle\mathcal{O}\rangle = \frac{\int d\psi d\bar\psi\, \mathcal{O}\, e^{iS}}{\int d\psi d\bar\psi\, e^{iS}} \tag{2.25}$$

and must be independent of our choice of integration variable in the numerator, which means that its functional derivative with respect to $\alpha(x)$ at $\alpha(x) = 0$ must vanish:

$$\left.\frac{\delta\langle\mathcal{O}\rangle}{\delta\alpha(x)}\right|_{\alpha=0} = 0\ . \tag{2.26}$$

If we compute this, ignoring the possibility of a Jacobian depending on α, we obtain the quantum version of Noether's theorem, namely a Ward Takahashi identity:

$$\partial_\mu\langle j^\mu_A(x)\,\mathcal{O}\rangle = 2im\langle\bar\psi\gamma_\chi\psi(x)\,\mathcal{O}\rangle\ , \tag{2.27}$$

and since this is true for any operator $\mathcal{O}$, this gives us eq. (2.11) as an operator equation.

However, following Fujikawa, we also get a phase from the Jacobian in the measure. In order to deal with convergence issues carefully, it behooves us to work in Euclidean spacetime, where the path integral is given by

$$\int d\psi\, d\bar\psi\, e^{-S_E}\ , \qquad S_E = \int d^2x\, \mathcal{L}_E \tag{2.28}$$

where the Euclidean Lagrange density for a Dirac fermion is given by

$$\mathcal{L}_E = \bar\psi\left(\not{D} + m\right)\psi\ , \qquad \{\gamma^\mu_E, \gamma^\nu_E\} = 2\delta_{\mu\nu}\ , \tag{2.29}$$

where the Euclidean and Minkowski gamma matrices are related by $\gamma^0_E = \gamma^0_M$ and $\gamma^1_E = -i\gamma^1_M$. (There is no distinction between upper and lower indices in Euclidean spacetime, although I may use upper and lower indices for typesetting reasons). Note that the γ^μ_E matrices are all hermitian, while $D_\mu = (\partial_\mu - iqA_\mu)$ is anti-hermitian. Now if we repeat the above steps, considering a chiral transformation while ignoring a possible Jacobian, we obtain the Euclidean version of eq. (2.27), namely

$$S_E \to S_E + \int d^2x \left[i\partial_\mu\alpha(x)\bar{\psi}\gamma_\mu\gamma_\chi\psi + 2im\alpha(x)\bar{\psi}\gamma_\chi\psi \right] \ , \tag{2.30}$$

and therefore

$$\partial_\mu \langle j^\mu_A(x)\,\mathcal{O}\rangle = 2m\langle\bar{\psi}\gamma_\chi\psi(x)\,\mathcal{O}\rangle \qquad \text{(Euclidean)} \ . \tag{2.31}$$

To compute the Jacobian of the chiral transformation, let's expand

$$\psi(x) = \sum_n c_n\phi_n(x) \ , \qquad \bar{\psi} = \sum_n \bar{c}_n\phi^\dagger_n(x) \ , \tag{2.32}$$

where c_n and $\bar{c}_n$ are Grassmann numbers and the $\phi_n(x)$ are a complete set of functions. The integration measure then becomes

$$d\psi d\bar{\psi} = \prod_m dc_m \prod_n d\bar{c}_n \ . \tag{2.33}$$

Under our transformation eq. (2.23) we have $\psi(x) \to \sum_n c'_n\phi_n(x)$ and $\bar{\psi} \to \sum_n \bar{c}'_n\phi^\dagger_n(x)$, where to $O(\alpha)$ we have

$$\begin{aligned} c'_n &= c_n + i\sum_m c_m \int d^2x\,\alpha(x)\phi^\dagger_m(x)\gamma_\chi\phi_n(x) \equiv c_n + i\sum_m c_m \langle m|\alpha\gamma_\chi|n\rangle \ , \\ \bar{c}'_n &= \bar{c}_n + i\sum_m \bar{c}_m \langle n|\alpha\gamma_\chi|m\rangle \ . \end{aligned} \tag{2.34}$$

Note that $\langle m|\alpha\gamma_\chi|n\rangle = \langle n|\alpha\gamma_\chi|m\rangle$ since $\alpha(x)\gamma_\chi$ is Hermitian. Using the relation $\det(1+\epsilon) = 1 + \mathrm{Tr}\,\epsilon + O(\epsilon^2)$, the Jacobian of this transformation can be computed and one finds that the measure changes by

$$d\psi d\bar{\psi} \to d\psi d\bar{\psi}\left|\frac{\partial c}{\partial c'}\right|\left|\frac{\partial \bar{c}}{\partial \bar{c}'}\right| = d\psi d\bar{\psi} e^{-2i\int d^2x\alpha\,\mathcal{A}} \ , \quad \mathcal{A}(x) = \sum_n \phi^\dagger_n(x)\gamma_\chi\phi_n(x) \ . \tag{2.35}$$

This has the effect of adding an extra piece $2i\alpha(x)\mathcal{A}(x)$ to the transformation of the Lagrangian in eq. (2.24) and is the source of the anomaly. So we need to compute $\mathcal{A}$.

That infinite sum in the definition of $\mathcal{A}$ is poorly defined, and so we need to regulate it, and we want to do so in a gauge invariant way. A convenient way is to choose the $\phi_n(x)$ functions to be eigenfunctions of $\not{D}$,

$$\not{D}\phi_n = i\lambda_n\phi_n \ . \tag{2.36}$$

Since $\not{D}$ is anti-hermitian, the λ_n eigenvalues are real. Furthermore, since under a gauge transformation $\phi_n(x) \to U(x)\phi_n(x)$ and $\not{D} \to U(x)\not{D}U^\dagger(x)$, one sees that the λ_n are gauge invariant. So a nice, gauge-invariant way to regulate our sum in $\mathcal{A}$ is to write

$$\mathcal{A}(x) = \lim_{\Lambda\to\infty}\sum_n e^{-\lambda_n^2/\Lambda^2}\,\phi_n^\dagger(x)\gamma_\chi\phi_n(x) = \sum_n \phi_n^\dagger(x)\gamma_\chi e^{\not{D}^2/\Lambda^2}\phi_n(x) \ , \tag{2.37}$$

where $\mathrm{Tr}_{\mathcal{H}}$ indicates a Hilbert space trace. We can write

$$\not{D}^2 = D_\mu D_\mu + qF_{\mu\nu}\sigma_{\mu\nu} = \partial_\mu\partial_\mu - iq(\partial_\mu A_\mu + 2A_\mu\partial_\mu) - q^2A_\mu A_\mu + +qF_{\mu\nu}\sigma_{\mu\nu} \ , \tag{2.38}$$

and to evaluate the sum it is convenient to change to a plane wave basis, so that

$$\mathcal{A}(x) = \lim_{\Lambda\to\infty}\mathrm{Tr}\ \gamma_\chi e^{qF_{\mu\nu}\sigma_{\mu\nu}/\Lambda^2}\int\frac{d^2k}{(2\pi)^2}e^{-ikx}e^{D^2/\Lambda^2}e^{ikx} \ . \tag{2.39}$$

In order to make the trace over Dirac indices not vanish, we need at least one power of $\sigma_{\mu\nu}$, which necessarily comes with a factor of $1/\Lambda^2$ from expanding the $e^{qF_{\mu\nu}\sigma_{\mu\nu}/\Lambda^2}$ factor, and the term linear in $\sigma_{\mu\nu}/\Lambda^2$ will be the leading term as $\Lambda\to\infty$. By dimensional analysis, we can get a factor of Λ^2 from the k integral, but we need only take the leading term, obtained by taking $A_\mu = 0$ in $D_\mu D_\mu$. Thus the only part of our expression that survives the Dirac trace and the $\Lambda\to\infty$ limit is

$$\begin{aligned}\mathcal{A}(x) &= \lim_{\Lambda\to\infty}\frac{qF_{\mu\nu}(x)}{\Lambda^2}\int\frac{d^2k}{(2\pi)^2}e^{-k^2/\Lambda^2}\,\mathrm{Tr}\ \gamma_\chi\sigma_{\mu\nu}\\ &= \lim_{\Lambda\to\infty}\epsilon_{\mu\nu}\frac{qF_{\mu\nu}(x)}{\Lambda^2}\left(\frac{\Lambda^2}{4\pi}\right)\\ &= \epsilon_{\mu\nu}\frac{qF_{\mu\nu}(x)}{4\pi} \ .\end{aligned} \tag{2.40}$$

When combining the above result with eq. (2.35) and differentiating with respect to $\alpha(x)$, we arrive at the correct version of eq. (2.31), namely

$$\begin{aligned}S_E &\to S_E + \int d^2x\,\left[i\partial_\mu\alpha(x)\bar\psi\gamma_\mu\gamma_\chi\psi + 2im\alpha(x)\bar\psi\gamma_\chi\psi + 2i\alpha(x)\mathcal{A}(x)\right] \ ,\\ \partial_\mu j_A^\mu(x) &= 2m\bar\psi\gamma_\chi\psi(x) + \frac{q}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x) \qquad \text{Euclidean} \ ,\end{aligned} \tag{2.41}$$

which on continuation back to Minkowski spacetime agrees with our previous results in eq. (2.16) and eq. (2.22).

2.5 The Atiyah-Singer index theorem

We can derive an interesting relationship between the spectrum of the Euclidean Dirac operator and the topology of background gauge fields. Note that since $\{\gamma_\chi, \not{D}\} = 0$, if follows that for nonzero eigenvalue λ_n

$$\not{D}\,(\gamma_\chi \phi_n) = -i\lambda_n\,(\gamma_\chi\phi_n)\,, \tag{2.42}$$

and so the nionzero eigenvalues come in $\pm\lambda_n$ pairs. For "zeromodes" (the case $\lambda_n = 0$) we can take ϕ_n to be an eigenfunction of γ_χ with eigenvalue ±1, and there is no reason for the number of RH (n_+) and the number of LH (n_-) zeromodes, to be the same. Note that in a chiral basis for the gamma matrices, for which $\gamma_\chi = \sigma_3$, we can write

$$\not{D} = \begin{pmatrix} 0 & -d^\dagger \\ d & 0 \end{pmatrix} \tag{2.43}$$

where d is a differential operator involving gauge fields. So the quantity $(n_+ - n_-)$ is the number of zero eigenfunctions of d, minus the number of zero eigenfunctions of $d^\dagger$, which mathematicians call the "index of d", and which physicists in an abuse of language often call the index of the Dirac operator. If d were a finite matrix, then necessarily its index would vanish—since a matrix M and its hermitian conjugate $M^\dagger$ have the same number of zero eigenvalues—but not so for a differential operator.

Note that for a given n corresponding to $\lambda_n \neq 0$, in the 2-dimensional subspace of the Hilbert space spanned by ϕ_n and $\gamma_\chi\phi_n$, γ_χ as a Hilbert space operator is just the matrix σ_1, which is traceless. Thus the index of d is given by $\text{ind}(d) = \text{Tr}_{\mathcal{H}}\,\gamma_\chi$, where $\text{Tr}_{\mathcal{H}}$ means trace over Hilbert space (as opposed to a trace over Dirac indices). But from eq. (2.35) and our result in eq. (2.40) we have

$$\begin{aligned} \text{ind}(d) &= \text{Tr}\ \gamma_\chi = (n_+ - n_-) \\ &= \int d^2x \sum_n \phi_n^\dagger(x)\gamma_\chi\phi_n(x) = \int d^2x\,\mathcal{A}(x) = \frac{q}{4\pi}\int d^2x\,\epsilon_{\mu\nu}F_{\mu\nu}(x)\ . \end{aligned} \tag{2.44}$$

The index is only defined in Euclidean spacetime.

2.6 Weyl fermions, chiral gauge theories, and gauge anomalies

A Dirac fermion can be thought of as a RH plus a LH Weyl fermion, and the Feynman diagram we computed for j_A^μ is just the sum of contributions from each. Under the $U(1)_V$ current that we gauge, both of these Weyl fermions have charge q, while under the $U(1)_A$ symmetry they have charges +1 and −1 respectively. Thus the factor of q appearing in the anomaly term can be replaced by

$$q \to \frac{\tilde{q}_R q_R - \tilde{q}_L q_L}{2} \tag{2.45}$$

where in the case we looked at, $q_R = q_L = q$ and $\tilde{q}_R = -\tilde{q}_L = 1$. However, we can more generally consider a theory with N_R RH Weyl fermions and N_L LH Weyl fermions and compute the the anomaly of a current

$$\tilde{j}^\mu = \bar{\psi}_R \tilde{Q}_R \gamma^\mu \psi_R + \bar{\psi} \tilde{Q}_L \gamma^\mu \psi_L \ , \tag{2.46}$$

where $\tilde{Q}_{R,L}$ are diagonal N_R and N_L dimension matrices respectively, in the presence of a U(1) gauge field A_μ coupling to the current

$$j^\mu = \bar{\psi}_R Q_R \gamma^\mu \psi_R + \bar{\psi} Q_L \gamma^\mu \psi_L \ . \tag{2.47}$$

Then using an obvious generalization of the formula in eq. (2.45) we would find the anomaly

$$\partial_\mu \tilde{j}^\mu = \frac{1}{4\pi} \left(\mathrm{Tr}\, \left[\tilde{Q}_R Q_R \right] - \mathrm{Tr}\, \left[\tilde{Q}_L Q_L \right] \right) \epsilon_{\mu\nu} F^{\mu\nu} \equiv \frac{1}{4\pi} \mathrm{Tr}\, \tilde{Q} Q \Big|_L^R \epsilon_{\mu\nu} F^{\mu\nu} \ . \tag{2.48}$$

Even more generally, we can consider the case of non-abelian symmetries and gauge fields, and find

$$\partial_\mu \tilde{j}^\mu_b = \frac{1}{4\pi} \mathrm{Tr}\, \left[\tilde{Q}^b Q^a \right] \Big|_L^R \epsilon_{\mu\nu} F^{\mu\nu}_a \ , \tag{2.49}$$

where a is summed over the generators of the gauge group, and the $F^{\mu\nu}_a$ are the non-abelian field strength tensors, where the $\tilde{Q}^\alpha$ matrices commute with all of the Q^a matrices, so that $\tilde{j}^\mu_\alpha$ is classically conserved.

However, something is sick about the theory if we find a nonzero anomalous divergence for the gauge current itself, which would break gauge invariance, given by the above expression with the tildes removed. This would be what is known as a "gauge anomaly" and it must vanish in a sensible theory, which gives us the constraint that the trace must vanish, namely

$$\mathrm{Tr}\, Q^b Q^a \Big|_L^R = 0. \tag{2.50}$$

Clearly this will hold for any theory which has equal numbers of LH and RH Weyl fermions which have identical gauge charges. Such a theory is called "vector-like", and both QED and QCD are examples, but not the Standard Model. Gauge theories that are no vector-like are called chiral gauge theories. An example of a sick nonabelian chiral gauge theory is $SU(3)$ with a single RH Weyl fermion transforming as a 3; for this theory Tr $Q^b Q^a\Big|_L^R = (\frac{1}{2}\delta_{ab} - 0) \neq 0$, which means that the gauge anomalies do not cancel and the theory is sick. An example of a healthy Abelian chiral gauge theory is one with a U(1) gauge group with two RH Weyl fermions and one LH Weyl fermion with the charges

$$Q_R = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} , \qquad Q_L = 5 , \tag{2.51}$$

for which

$$\text{Tr } Q_R^2 - \text{Tr } Q_L^2 = 3^2 + 4^2 - 5^2 = 0 . \tag{2.52}$$

This is an example of a theory without gauge anomalies that cannot be written in terms of complete Dirac fermions. Note that it is not possible to write gauge invariant mass terms for the fermions without spontaneously breaking the gauge symmetry – this is the central characteristic of chiral gauge theories (like the standard model).

3 The chiral anomaly in 3+1 dimensions

3.1 The $U(1)_A$ anomaly

An analogous violation of the $U(1)_A$ current occurs in $3+1$ dimensions as well. When the number of spacetime dimensions is even one can always define a matrix γ_χ which anti-commutes with all of the γ_μ – and hence the Lorentz generators $\sigma_{\mu\nu}$, allowing one to reduce a Dirac spinor into a RH plus a LH Weyl spinor, and one can similarly define an axial rotation and axial current. And as in 1+1 dimensions, this $U(1)_A$ current has an anomalous divergence when the $U(1)_V$ current is gauged. One might guess that the analogue of $\epsilon_{\mu\nu}F^{\mu\nu} = 2E$ in the anomalous divergence eq. (2.16) would be the quantity $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 8\vec{E}\cdot\vec{B}$, which has the same dimensions as the divergence of a current and the same properties under parity and time reversal. So we should consider the behavior a massless Dirac fermion in $(3+1)$ in parallel constant E and B fields. First turn on a B field pointing in the $\hat{z}$ direction: this gives rise to Landau levels, with energy levels E_n characterized by non-negative integers n as well as spin in the $\hat{z}$ direction S_z and momentum p_z, where

$$E_n^2 = p_z^2 + (2n+1)qB - 2qBS_z . \tag{3.1}$$

The dispersion relation looks like that of an infinite tower of one-dimensional fermions with momentum p_z and mass $m_{n,\pm}$, where

$$m_{n\pm}^2 = (2n+1)qB - 2qBS_z \ , \quad S_z = \pm\tfrac{1}{2} \ . \tag{3.2}$$

The state with $n = 0$ and $S_z = +\frac{1}{2}$ is distinguished by having $m_{n,+} = 0$; it behaves like a *massless* one-dimensional Dirac fermion moving along the $\hat{z}$ axis with dispersion relation $E = |p_z|$. If we now turn on an electric field also pointing along the $\hat{z}$ direction we know what to expect from our analysis in $1+1$ dimensions: we find an anomalous divergence of the axial current equal to $g_0\, qE/\pi$ where g_0 is the transverse density of states in the $n = 0$ and $S_z = +\frac{1}{2}$ state.

How do we compute g_0? We need to recover the free fermion result as $B \to 0$. For free fermions we have the density of states is $d^3p/(2\pi)^3 = (dp_z/2\pi)(dp_x dp_y/(2\pi)^2)$, where the second factor is the transverse density of states. If we convert to radial coordinates and integrate over angle in the $x-y$ plane we are left with a density of states $g = p_\perp dp_\perp/(2\pi)$, where $E = \sqrt{p_z^2 + p_\perp^2}$. If we equate this formula with eq. (3.1) we get

$$p_\perp^2 = [(2n+1)qB - 2qBS_z] \qquad \Longrightarrow \qquad 2p_\perp dp_\perp = 2qB \ , \tag{3.3}$$

where $2p_\perp dp_\perp = d(p_\perp^2)$ is computed by seeing how $p_\perp^2$ in eq. (3.3) changes as we change n to $n+1$. Therefore the transverse density of states of the n^{th} Landau level is independent of n and given by

$$g_n = \frac{dp_\perp dp_\perp}{2\pi} = \frac{qB}{2\pi} \ . \tag{3.4}$$

This allows us to conclude that for a massless fermion in 3+1 dimensions the anomaly is

$$\partial_\mu j_A^\mu = g_0 \frac{qE}{\pi} = \frac{q^2}{2\pi^2} EB = \frac{q^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \ . \tag{3.5}$$

If we include explicit breaking from the fermion mass term, we get

$$\partial_\mu j_A^\mu = 2im\overline{\psi}\gamma_\chi\psi + \left(\frac{q^2}{16\pi^2}\right) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \ . \tag{3.6}$$

One can also derive this result by computing Feynman diagrams with a Pauli-Villars regulator as in the $1+1$ dimensional example; now the relevant graph is the triangle diagram of Fig. 4. And one can also derive the result using Fujikawa's method where the $F\tilde{F}$ structure arises because Tr $\gamma_\chi =$ Tr $\gamma_\chi \sigma_{\mu\nu} = 0$ in 3+1 dimensions, but Tr $\gamma_\chi \sigma_{\mu\nu} \sigma_{\alpha\beta} \propto \epsilon_{\mu\nu\alpha\beta}$. The two powers of $F_{\mu\nu}\sigma_{\mu\nu}$ are accompanied by a factor of $1/\Lambda^4$,

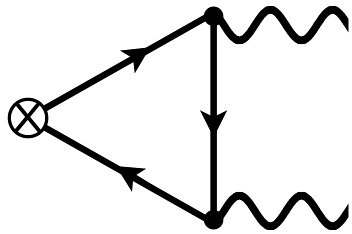

Figure 4. *The $U(1)_A$ anomaly diagram in 3+1 dimensions, with insertions of $\overline{\psi}\gamma_\chi\psi$ or $\overline{\psi}\gamma^\mu\gamma_\chi\psi$.*

which is canceled by the factor of Λ^4 arising from the 4-dimensional Gaussian integral over k.

If the gauge fields are nonabelian, the analogue of eq. (3.6) is

$$\partial_\mu j^\mu_A = 2im\overline{\psi}\gamma_\chi\psi + \left(\frac{g^2}{16\pi^2}\right)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}_a F^{\rho\sigma}_b \,\mathrm{Tr}\; T_a T_b \; . \tag{3.7}$$

If the fermions transform in the defining representation of $SU(N)$, it is conventional to normalize the coupling g so that $\mathrm{Tr}\; T_a T_b = \frac{1}{2}\delta_{ab}$. This is still called an "Abelian anomaly", since j^μ_A generates a $U(1)$ symmetry.

As in 1+1 dimensions, we can generalize the anomaly equation theory to account for multiple flavors of both RH and LH Weyl fermions, and consider the anomalous divergence of arbitrary nonabelian currents. These are proportional to the group theory factor

$$\mathrm{Tr}\;\left(\{T_a, T_b\}\tilde{Q}\right)\Bigg|^{RH}_{LH} \; , \tag{3.8}$$

where j^μ_A has been replaced with the current $\bar{\psi}_R\tilde{Q}_R\psi_R + \bar{\psi}_L\tilde{Q}_L\psi_L$, where I have suppressed the N_R and N_L flavor indices for the RH and LH Weyl fermions. The gauge currents have the same form, with $\tilde{Q}_{R,L}$ replaced by $T_{a,R}$ and $T_{a,L}$.

Insisting that the gauged currents themselves all be divergenceless, e.g. that the gauge anomalies all cancel, is equivalent to the group theory statement found by replacing $\tilde{Q}$ in the above equation by the T_a:

$$\mathrm{Tr}\;\left(\{T_a, T_b\}T_c\right)\Bigg|^{RH}_{LH} = 0 \; . \tag{3.9}$$

An example of a nontrivial chiral gauge theory with cancelling anomalies is $SU(5)$ with LH Weyl fermions transforming as a $\bar{5} \oplus 10$. Another example is the $SU(3) \times SU(2) \times U(1)$ Standard Model.

I encourage you to perform all of the analogous calculations in 3+1 dimensions that I did in 1+1 (the Fujikawa derivation of the anomaly, the Feynman diagram calculation, and calculation of the index). It is also an interesting exercise to show that the Standard Model with any number of families is free of gauge anomalies.

4 A parity anomaly

4.1 Fermion masses and parity in odd dimensions

In odd dimensions there is no analogue of γ_χ and therefore there is no such thing as chiral symmetry. Nevertheless, fermion masses still break a symmetry: parity. In a theory with parity symmetry one has extended the Lorentz group to include improper rotations: spatial rotations R for which the determinant of R is negative. Parity can be defined as a transformation where an odd number of the spatial coordinates flip sign. In even dimensions parity can be the transformation $\mathbf{x} \to -\mathbf{x}$ and

$$\psi(\mathbf{x},t) \to \gamma^0\psi(-\mathbf{x},t) \qquad (\text{parity, } d \text{ even}) \ . \tag{4.1}$$

The role of the γ^0 is to transform the kinetic term correctly to realize $\vec{x} \to -\vec{x}$:

$$\gamma^0(\partial_0\gamma^0)\gamma^0 = \partial_0\gamma^0 \ , \qquad \gamma^0(\nabla_i\gamma^i)\gamma^0 = -\nabla_i\gamma^i \ . \tag{4.2}$$

Since $\{\gamma^0, \gamma_\chi\} = 0$, ψ_L and ψ_R are exchanged under parity and a Dirac mass term is parity invariant.

However, in even space dimensions the transformation $\mathbf{x} \to -\mathbf{x}$ is just a proper rotation; instead we must define parity as the transformation which just flips the sign of one coordinate such as x^1 (or an odd number), and

$$\psi(\mathbf{x},t) \to \gamma^1\psi(\tilde{\mathbf{x}},t) \ , \qquad \bar{\psi}(\mathbf{x},t) \to -\bar{\psi}(\tilde{\mathbf{x}},t)\gamma^1 \ , \qquad \tilde{\mathbf{x}} = (-x^1, x^2, \ldots, x^{2k}) \tag{4.3}$$

since

$$-\gamma^1(\partial_\mu\gamma^\mu)\gamma^1 = \begin{cases} +\partial_\mu\gamma^\mu & \mu \neq 1 \\ -\partial_\mu\gamma^\mu & \mu = 1 \end{cases} \qquad (\text{no sum on } \mu) \ . \tag{4.4}$$

Remarkably, a Dirac mass term flips sign under parity in this case; and since there is no chiral symmetry in odd d to rotate the phase of the mass matrix, the sign of the quark mass has physical meaning. Note that it is still possible to define a parity invariant theory of massive fermions, however, provided that they come in pairs with masses $\pm M$, and parity is defined to interchange the two, while flipping the sign of M...so long as the regulator also respects parity.

4.2 The non-decoupling of parity violation

We have seen that fermion masses break chiral symmetry for even d and that they can break parity for odd d. One might assume then that a theory with a massless fermion in odd d, for which the action is parity invariant, might see anomalous parity violation arise from a massive regulator, such as a Pauli-Villars fermion with mass Λ. Indeed that occurs as can be seen both by power counting and explicit calculation.

In $2k+1$ spacetime dimensions one can write down an interesting operator, called the Chern Simons operator, which for an Abelian gauge field is proportional to

$$q^k \epsilon^{\alpha_1 \cdots \alpha_{2k+1}} A_{\alpha_1} F_{\alpha_2 \alpha_3} \cdots F_{\alpha_{2k} \alpha_{2k+1}} \ . \tag{4.5}$$

Note that this operator violates parity and time reversal. This operator is not gauge invariant, but transforms into a total derivative under a gauge transformation, so that its inclusion in the Lagrangian does not spoil the gauge invariance of the action. The Chern Simons form for nonabelian gauge fields is more complicated, and requires a quantized coefficient in order for the action to be gauge invariant. Let's count dimensions: with $D_\mu = (\partial_\mu - iqA_\mu)$ we have the mass dimension of qA_μ is 1, and therefore the mass dimension of the above operator is $2k+1 = d$. That means that its coefficient in the action must be dimensionless. On the other hand, if this is to be generated radiatively, its coefficient must be proportional to explicit parity symmetry violation. That suggests that on integrating out a Pauli-Villars fermion with mass M, this operator can and should be generated with a coefficient proportional to $\Lambda/|\Lambda|$, which flips sign when Λ flips sign, and which is dimensionless Such a coefficient arises naturally from an integral of the form

$$\int \frac{d^{2k+1}p}{(2\pi)^{2k+1}} \frac{\Lambda}{(p^2+\Lambda^2)^{2k}} \propto \frac{\Lambda}{|\Lambda|} \ , \tag{4.6}$$

Because the coefficient is proportional to $\Lambda/|\Lambda|$, a Chern-Simons operator generated radiatively by a Pauli-Villars field survives the limit $\Lambda \to \infty$. This effect is called a "parity anomaly" and arises because we have to break parity to regulate the theory.

For domain wall fermions we will be interested in a closely related but slightly different problem: understanding the low energy physics of a theory with a massive fermion. In this case bioth the massive fermion and the Pauli-Villars field contribute to a Chern Simons operator on integrating out a heavy fermion. In $2+1$ dimensions with an Abelian gauge the graph in Fig. 5 gives rise to the low-energy Lagrangian

$$\mathcal{L}_{CS} = -\frac{q^2}{4\pi} \left(\frac{m}{|m|} - \frac{\Lambda}{|\Lambda|} \right) \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \ . \tag{4.7}$$

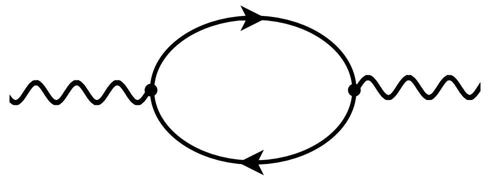

Figure 5. *Integrating out a heavy fermion in three dimensions gives rise to the Chern Simons term in the effective action of eq. (4.7).*

We will be interested in the case where the mass m is space dependent and changes sign. We see then that the contributions from the fermion and Pauli-Villars fields cancel in regions where the two have the same sign mass, but add in the region where they have opposite sign mass.

It is striking how related the parity and chiral anomalies look. First of all, they both involve an epsilon tensor. This is the hallmark of a topological quantity: in curved spacetime, and integration measure and epsilon tensor depend on the metric with $d^dx \to d^dx\sqrt{g}$ and $\epsilon \to \epsilon/\sqrt{g}$, where g is the determinant of the metric, and these cancel when integrating a quantity with an epsilon tensor such as the Chern-Simons operator, so that the result is independent of geometry and choice of coordinates. But beyond that similarity, suppose one differentiates $\mathcal{L}_{CS}$ in eq. (4.7) with respect to the gauge field to compute the current which contributes to Maxwell's equations. In $d = 2 + 1$, for example, one finds:

$$j_\mu = \frac{\partial \mathcal{L}_{CS}}{\partial A_\mu} = -\frac{q^2}{8\pi}\left(\frac{m}{|m|} - \frac{\Lambda}{|\Lambda|}\right)\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \tag{4.8}$$

where, if I pick $\mu = 2$, what I get is related to the chiral anomaly we found in eq. (2.16) for $1 + 1$ dimensions. This is a peculiar result, because it says that an electric field in the x^1 direction gives rise to a current in the x^2 direction. This phenomenon seems to be similar to the Hall effect, where time reversal violation occurs due to a magnetic field instead of a fermion mass. In fact, in the next section we will see that this theory describes phenomena identical to the Integer Quantum Hall Effect, and that the transverse conductivity σ_{xy} comes in quanta of h/q^2, where the quantization is topological in origin.

5 Anomalies, edge states, and topology

5.1 Motivation

The connection between the parity anomaly in odd spacetime dimensions and the chiral anomaly in one dimension lower can be understood better by considering spacetimes with boundaries, and the fermion states that live on those boundaries. The key feature of a boundary is that it beaks translational invariance, and so equivalently, rather than having a boundary, one can consider fermions in infinite spacetime with a space dependent mass. In particular we will consider a mass with a step function profile, going from positive on one side of space, and negative on the other.

To be concrete, consider a Dirac fermion in 2+1 dimensions with a mass that changes sign in one of the space directions. Curiously, one finds that bound to the line where the mass vanishes, there exists a massless chiral fermion. This is surprising because there is no anomaly that can prevent us from gauging fermion number in the 2+1 theory, yet we are not supposed to be able to gauge a single chiral fermion in 1+1, since the gauge charge is not conserved in a background electric field. The resolution is that the Chern-Simons current in the 2+1 bulk flows onto the 1+1 mass defect, so that charge is not actually violated. This is only possible because the Chern-Simons terms that arises from the parity anomaly in 2+1 is related to the chiral anomaly in 1+1.

To see how edge states arise, consider a fermion in consider spacetime with dimension $d+1$ dimensions, where the coordinates are written as $\{t, x_1, \ldots x_{d-1}, s\} \equiv \{x_\mu, s\}$, where $\mu = 0, \ldots, d-1$ and s is what I call the coordinate x_d. To avoid annoying factors of i, I will work in Euclidean spacetime for now, so the γ matrices are hermitian. This fermion is assumed to have an s-dependent mass with the simple form

$$m(s) = m\,\epsilon(s) = \begin{cases} +m & s > 0 \\ -m & s < 0 \end{cases} , \qquad m > 0 \ . \tag{5.1}$$

This mass function explicitly breaks the Poincaré symmetry of $d+1$ dimensional spacetime, but preserves the Poincaré symmetry of $(d-1)+1$ dimensional spacetime. The fermion is also assumed to interact with $(d-1)+1$-dimensional background gauge fields which are s-independent and have no s component. The Dirac equation may be written as:

$$\left[\not{D} + \gamma_d \partial_s + m(s)\right] \Psi(x_\mu, s) = 0 \ , \tag{5.2}$$

where $\not{D}$ is the covariant Dirac operator for $(d-1)+1$ dimensions. We can perform separation of variables and expand the spinor Ψ as the product of functions of s times spinors $\psi(x_\mu)$,

$$\Psi(x_\mu, s) = \sum_n \left[b_n(s)P_+ + f_n(s)P_-\right]\psi_n(x_\mu)\ , \qquad P_\pm = \frac{1\pm\gamma_d}{2}\ , \tag{5.3}$$

satisfying the eigenvalue equations

$$\begin{aligned}[\partial_s + m(s)]b_n(s) &= \mu_n f_n(s)\ ,\\ [-\partial_s + m(s)]f_n(s) &= \mu_n b_n(s)\ ,\end{aligned} \tag{5.4}$$

where the ψ_n are general functions of x_μ, which can later be expanded themselves in some useful basis. That we can write eq. (5.4) can motivated by finite matrices. Consider a non-hermitian matrix M; it will in general have different left- and right-eigenvectors. Let the unitary matrices L and R diagonalize the Hermitian matrices $MM^\dagger$ and $M^\dagger M$ respectively. For a finite matrix M, it is guaranteed that $MM^\dagger$ and $M^\dagger M$ have the same real non-negative eigenvalues μ_n^2. The matrices L and R are only determined up to diagonal phases, and it is possible to choose the phases of L and R such that the matrix $L^\dagger MR$ is diagonal, with entries equal to $|\mu_n|$. (This fact is used in the Standard Model when we diagonalize the quark masses, and $L^\dagger R$ plays the role of the CKM matrix, while the $|\mu_n|$ are the quark masses.) In the present case, the differential operator $(\partial_s + m(s))$ plays the role of M, while $(-\partial_s + m(s))$ plays the role of $M^\dagger$. Then the $b_n(s)$ and $f_n(s)$ functions are eigenstates of $M^\dagger M$ and $MM^\dagger$ respectively, and so we know that the normalizable b_n functions form a complete orthogonal basis, as do the f_n, justifying the expansion eq. (5.3).

An important difference between finite matrices and differential operators, however, is that $MM^\dagger$ and $M^\dagger M$ can have a different number of zero eigenvalues. One might expect all the eigenvalues in eq. (5.4) to satisfy $|\mu_n| \gtrsim O(m)$, since that is the only scale in the problem. However, there is also a solution to eq. (5.4) with eigenvalue $\mu = 0$ given by

$$b_0 = N e^{-\int_0^s m(s')ds'} = N e^{-m|s|}\ . \tag{5.5}$$

This solution is localized near the defect at $s = 0$, falling off exponentially fast away from it. Note that since $b_0(s)$ is time-independent, the existence of this solution is the same in Minkowski spacetime. There is no analogous solution to eq. (5.4) of the form

$$f_0 \sim e^{+\int_0^s m(s')ds'}\ ,$$

since that would be exponentially growing in $|s|$, not normalizable, and hence not in the Hilbert space. In terms of this expansion, the $2k+1$-dimensional Dirac action can be written as an infinite sum of $2k$-dimensional Dirac actions:

$$S = \int d^d x \int ds\, \bar\Psi\left(\not{D} + \gamma_d\partial_s + m\epsilon(s)\right)\Psi$$

$$= \sum_{k,\ell} \int d^d x \int ds\, \bar\psi_k(x) \left[b_k^\dagger(s) P_- + f_n^\dagger(s) P_+\right] \left(\not{D} + \gamma_d \partial_s + m\epsilon(s)\right) \left[b_\ell(s) P_+ + f_\ell(s) P_-\right] \psi_\ell(x)$$

$$= \int d^d x \left[\bar\psi_0 \not{D} P_+ \psi_0 + \sum_{k\neq 0} \bar\psi_k \left(\not{D} + \mu_n\right) \psi_k\right] \tag{5.6}$$

So we see that the spectrum consists of a single massless fermion which is an eigenstate of γ_d with eigenvalue $+1$, and an infinite tower of massive fermions with mass $O(m)$ and higher. The massless fermion is localized at the defect at $s = 0$, whose profile in the transverse extra dimension is given by eq. (5.5); the massive fermions are not localized in the extra dimension. Because of the gap in the spectrum, at low energy the accessible part of the spectrum consists only of the massless edge state. Since the states in the bulk are gapped, this is what would be referred to as an insulator by condensed matter physicists. However, we will see that the bulk is characterized by different topological phases, so in fact this is an example of a *topological insulator* [1, 2].

Now specialize to $d = 2$ so that the bulk is $2+1$ dimensional and the surface where M changes sign is $1+1$ dimensional. The matrix $\gamma_d = \gamma_2$ is equal to γ_χ of the $1+1$ dimension world, and so the edge state bound to the surface where m changes sign is a RH chiral fermion.

Some comments are in order:

- It is not a paradox that the low energy theory of a single right-handed chiral fermion ψ_0 violates parity in $1+1$ dimensions, since the mass for Ψ breaks parity in $2+1$;
- Furthermore, nothing is special about right-handed fermions, and a left handed mode would have resulted if we had chosen the opposite sign for the mass in eq. (5.1).
- The fact that a chiral mode appeared at all is a consequence of the normalizability of $\exp(-\int_0^s m(s')ds')$, which in turn follows from the two limits $m(\pm\infty)$ being nonzero with opposite signs. Any function $m(s)$ with that boundary condition will support a single chiral mode, although in general there may also be a number of very light fermions localized in regions wherever $|m(s)|$ is small — possibly extremely light if $m(s)$ crosses zero a number of times, so that there are widely separated defects and anti-defects.
- When dynamical gauge fields are included, gauge boson loops will generate contributions to the fermion mass function which are even in s. If the coupling is sufficiently weak, it cannot effect the masslessness of the chiral mode. However

if the gauge coupling is strong, or if the mass m is much below the cutoff of the theory, the radiative corrections could cause the fermion mass function to never change sign, and the chiral mode would not exist. Or it could still change sign, but become small in magnitude in places, causing the chiral mode to significantly delocalize. An effect like this can cause trouble with lattice simulations using domain wall fermions at finite volume and lattice spacing.

5.2 The Callan-Harvey mechanism

Now turn on background gauge fields and see how the anomaly works, following [3]. To do this, I integrate out the heavy modes in the presence of a background gauge field. Although I will be interested in having purely $2k$-dimensional gauge fields in the theory, I will for now let them be arbitrary $2k+1$ dimensional fields. Since it is hard to integrate out the heavy modes exactly, because of their complicate $b_n(s)$ and $f_n(s)$ wave functions, I will perform the calculation as if the mass m was constant, and then substitute $m(s)$; this is not valid where $m(s)$ is changing rapidly (near the domain wall) but should be adequate farther away. Also — in departure from the work of [3], we will include a Pauli-Villars field with constant mass $\Lambda < 0$, independent of s; this is necessary to regulate fermion loops in the wave function renormalization for the gauge fields, for example.

When one integrates out the heavy fields in Minkowski spacetime, specializing to $2+1$ dimensions, one generates a Chern Simons operator in the effective Lagrangian, as discussed in §4.2 [4] :

$$\mathcal{L}_{\text{CS}} = -\frac{q^2}{8\pi}\left(\frac{m(s)}{|m(s)|} - \frac{\Lambda}{|\Lambda|}\right)\epsilon^{abc}A_a\partial_b A_c = -\frac{q^2}{8\pi}\left(\epsilon(s)-1\right)\epsilon^{abc}A_a\partial_b A_c \ . \tag{5.7}$$

You should not trust the $\epsilon(s)$ function near $s=0$, but the fact that it equals ± 1 at $s = \pm\infty$ is reliable. Note that with $\Lambda < 0$, the coefficient of the operator equals 0 on the side where $m(s)$ is positive, and equals -2 on the side where it is negative. Differentiating $\mathcal{L}_{\text{CS}}$ by A_μ gives the bulk current:

$$j^a_{\text{CS}} = -\frac{q^2}{8\pi}\left(\epsilon(s)-1\right)\epsilon^{abc}F_{bc} \tag{5.8}$$

So when we turn on background $2k$ dimensional gauge fields, particle current flows either onto or off of the domain wall along the transverse s direction on the right side (where $m(s) = +m$).

[4] One has to be careful to choose the signs of γ^d in the $d+1$ theory and γ_χ in the theory on the domain wall so that they are consistent.

This current $j^a_{\rm CS}$ does not account for the zeromode contribution. The zeromode is a RH Weyl fermion, and since a RH current may be written as

$$j^\mu_R = \tfrac{1}{2}\left(j^\mu_V + j^\mu_A\right) \ , \tag{5.9}$$

its anomalous divergence (from eq. (2.16)) is

$$\partial_\mu j^\mu_R = \tfrac{1}{2}\partial_\mu j^\mu_A = \frac{q^2}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} \ . \tag{5.10}$$

Thus the divergence of total current in our theory – the divergence of the bulk current plus that of the zeromode current – is

$$\partial_a J^a_{\rm tot} = \partial_s j^2_{\rm CS} + \partial_\mu j^\mu_{\rm zm} = -\frac{q^2}{4\pi}\delta(s)\epsilon^{\mu\nu}F_{\mu\nu} + \frac{q^2}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} = 0 \ . \tag{5.11}$$

So we see that as charge builds up on the domain wall, it flows off into the bulk, and overall the 2+1 theory's current is conserved[5].

Why does this current in the bulk seem bizarre? Because the spectrum in the bulk is gapped – there are no light excitations there. In particle physics we are accustomed to the fact that if we want to see heavy particles we have to spend billions of dollars! Yet in the bulk we can generate currents with very cheap (weak, smooth) gauge fields...how is this possible? Furthermore, the currents are transverse to the applied electric field! Well, you have hopefully seen this before: the phenomenon is identical to the Integer Quantum Hall effect. In that case the UV theory is quite different (magnetic field, nonrelativistic electrons, Landau levels) but the underlying topological phase structure and description in terms of Chern-Simons currents is identical. And where the condensed matter system relies on the B field to provide P and T violation, here the Ψ mass m performs the same role. More on this connection later.

Where is this charge coming from? In this model, from $x_2 = \infty$. If we worked with space compactified in the x_2 direction, then there would be both a domain wall with a RH Weyl edge state, and and anti-domain wall with a LH edge state, the fermion number on one side would be depleted and on the other augmented, with a current flowing between them.

This analysis works for a $3+1$ dimensional chiral zeromode stuck to a domain wall embedded in $4+1$ dimensions. In this case the Chern-Simons operaotr is proportional to $\epsilon A\partial A\partial A$ and the anomaly is proportional to $\epsilon\partial A\partial A$.

[5] I said to distrust the $\epsilon(s)$ function near $s = 0$, and now I used $\partial_s\epsilon(s) = 2\delta(s)$! What one can reliably show is that the total charge flowing out at $s = -\infty$ is equal to the charge created on the domain wall by the anomaly, by integrating $\int_{-s_0}^{s_0}\partial j ds$ over a slab of width $2s_0 \gg 1/m$ that contains the domain wall and the edge states.

5.3 The role of topology

Note that we can compute the conductivity σ_{xy}, given by $|j_2/E_1| = q/2\pi$. This will look more familiar to a condensed matter physicist if we recall that we have set $\hbar = 1$, so that this really is $\sigma_{xy} = q/h$, the famous quantum of conductivity discovered by Von Klitzing. At this point, however, we do not see an analog of the integer jumps in σ_{xy} as a function of the magnetic field that he discovered, and so our result doesn't look obviously "quantized" – but it does when the theory is formulated on a lattice.

My colleague David Thouless received the Nobel Prize for showing that the quantization of current in the Integer Quantum Hall Effect arises from topology. Note that this arises in condensed matter systems where electrons hop on a lattice. Topology is a property of smooth manifolds... how can one have topology on a lattice? The answer is that the topology is hiding in momentum space, not coordinate space. The same phenomenon occurs with the coefficient of the Chern Simons operator arising from the Feynman diagram of Fig. 5... but one has to go to Euclidean spacetime to see it.

With the admittedly awkward mostly-minus metric convention I am using, one can show that the following conversion holds:

$$\mathcal{L}_M = \bar{\psi}(i\not{D} - m)\psi + g_{\rm CS}\epsilon^{abc}A_a\partial_b A_c \implies \mathcal{L}_E = \bar{\psi}(\not{D} + m)\psi - ig_{\rm CS}\epsilon_{abc}A_a\partial_b A_c \; . \tag{5.12}$$

To compute the diagram in Fig. 5 in Euclidean spacetime we use the fermion propagator $S(p) = (i\not{p} + m)^{-1}$ and gauge field coupling $i\Lambda_\mu = iq\gamma_\mu = q\partial_\mu S^{-1}(p)$, where the derivative of the inverse propagator is with respect to momentum, $\partial_\mu \equiv \partial/\partial p_\mu$. Relating the gauge field coupling to the momentum derivative of the inverse fermion propagator is an expression of the Ward-Takahashi identity, and simply reflects the fact that the photon coupling and fermion propagator are related by gauge invariance, since derivatives acting on the fermions always come packaged as covariant derivatives, $D_\mu\psi$, with $D_\mu = \partial_\mu - iqA_\mu$. Using the Ward identity to compute the Feynman diagram in Fig. 5 may seem like overkill for free fermions, but in fact it is extremely helpful here because (i) it will allow us to see the role of topology, and (ii) the calculation can be immediately applied to the lattice where fermion propagators are much more complicated functions of momentum [4].

So we can compute use the Feynman diagram to compute the coupling $g_{\rm CS}$ that we get from integrating out a fermion of mass m. Since we only want $g_{\rm CS}$ we can expand to linear order in the momentum p flowing through the diagram, using the Fourier transform $p_\mu \leftrightarrow -i\partial/\partial x_\mu$ for momentum flowing out of a vertex, and can pick out the piece proportional to ϵ_{abc} by contracting the diagram with $\epsilon_{abc}/6$. The result is

$$g_{\rm CS} = (-1)\tfrac{1}{2}q^2\frac{\epsilon_{abc}}{6}\int\frac{d^3k}{(2\pi)^3}\,{\rm Tr}\,\left[\partial_a S^{-1}(k)\left(\partial_b S(p+k)|_{p=0}\right)\partial_c S^{-1}(k)S(k)\right].$$

$$(5.13)$$

In the above expression the factor of (-1) is from the fermion loop; the $1/2$ arises because the Chern-Simons term is quadratic in gauge fields. The factor of i from $\mathcal{L}_E$ and $-i$ from the Fourier transform $\partial_x \sim ip$ cancel. Remember that the derivatives in the integrand are with respect to k – the $\partial_{a,c}$ terms are from the photon couplings, using the Ward-Takahashi identity, and the ∂_b term is from our Taylor expansion of the graph to linear order in p_b. Then using the identity $\partial_b S = -S\partial_b S^{-1} S$, our expression can be rewritten as

$$g_{\rm CS} = \frac{q^2}{2}\frac{\epsilon_{abc}}{6}\int \frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\,\left[\left(S\partial_a S^{-1}\right)\left(S\partial_b S^{-1}\right)\left(S\partial_c S^{-1}\right)\right] . \tag{5.14}$$

A remarkable thing about the above expression is it is unchanged if we replace everywhere

$$S^{-1} \to U\frac{S^{-1}}{\sqrt{\det S^{-1}}} , \qquad U^{-1} = S\sqrt{\det S^{-1}} \tag{5.15}$$

so that

$$g_{\rm CS} = \frac{q^2}{2}\frac{\epsilon_{abc}}{6}\int \frac{d^3k}{(2\pi)^3}\,\mathrm{Tr}\,\left[\left(U^{-1}\partial_a U\right)\left(U^{-1}\partial_b U\right)\left(U^{-1}\partial_c U\right)\right] . \tag{5.16}$$

For our Dirac fermion,

$$U = \frac{m}{\sqrt{k^2+m^2}} + ik_\alpha\gamma_\alpha\frac{1}{\sqrt{k^2+m^2}} . \tag{5.17}$$

Note that U has the form

$$U = \cos\frac{\theta}{2} + i\hat{\theta}_i\gamma_i\sin\frac{\theta}{2} = e^{i\theta_i\gamma_i/2} , \tag{5.18}$$

with

$$\cos\frac{\theta}{2} = \frac{m}{\sqrt{k^2+m^2}} , \qquad \sin\frac{\theta}{2} = \frac{k}{\sqrt{k^2+m^2}} , \qquad \hat{\theta} = \hat{k} . \tag{5.19}$$

We can recognize U as an SU(2) matrix, and that the $\vec{\theta}$ variables parametrize a general SU(2) matrix as a point in a 3-ball of radius 2π with all points on the outside surface identified. This in turn is a parametrization of S^3, the 3-sphere (just like a disk with the outside edge identified parametrizes a 2-sphere).

So our Feynman integral represents a map from momentum space to SU(2)... but unfortunately, momentum space is not compact, and so we do not have a winding number interpretation. Note that when we integrate over all momenta k, $\cos\frac{\theta}{2} = m/\sqrt{k^2+m^2}$ remains positive, so we only visit half of the SU(2) ball when we integrate over all k. But this is where the Pauli-Villars regulator comes in. We can replace everywhere $S \to S/S_{\rm PV}$, as is evident from eq. (2.18), where in $S_{\rm PV}$ we replace $m \to \Lambda$, which means we replace $U \to U_{\rm reg} = U U_{\rm PV}^{-1}$:

$$\begin{aligned} U_{\rm reg} &= \frac{k^2 + m\Lambda}{\sqrt{(k^2+m^2)(k^2+\Lambda^2)}} + i k_\alpha \gamma_\alpha \frac{\Lambda - m}{\sqrt{(k^2+m^2)(k^2+\Lambda^2)}} \\ &\equiv \cos\frac{\theta_{\rm reg}}{2} + i\hat{\theta}_{{\rm reg},i}\gamma_i \sin\frac{\theta_{\rm reg}}{2} \ . \end{aligned} \tag{5.20}$$

Note that $U_{\rm reg}$ is again an SU(2) matrix, and that at $k=\infty$, $U_{\rm reg}=1$, while at $k=0$ we see that $U_{\rm reg}=1$ if $m\Lambda>0$, but $U_{\rm reg}=-1$ if $m\Lambda<0$. We have effectively compactified momentum space to a 3-sphere (we get the same $U_{\rm reg}$ no matter which direction we go to $k=\infty$) and the map from the 3-sphere to SU(2) is trivial of m and Λ have the same sign, and nontrivial if they have opposite sign. When we perform the integral in eq. (5.14) then we find

$$g_{\rm CS} = \frac{q^2}{4\pi}\nu \ , \tag{5.21}$$

where ν is the winding number of the map from the 3-sphere to SU(2) (also a 3-sphere), characterized by there homotopy group $\pi_3(SU(2)) = \mathbb{Z}$, with

$$\nu = -\tfrac{1}{2}\left(\frac{m}{|m|} - \frac{\Lambda}{|\Lambda|}\right) \ . \tag{5.22}$$

Plugging our value for $g_{\rm CS}$ into $\mathcal{L}_E$ of eq. (5.12), computing the current $j_\mu = i\partial\mathcal{L}_E/\partial A_\mu$ gives a result for the Hall current

$$j_a = g_{\rm CS}\epsilon_{abc}F_{bc} = \frac{q^2}{4\pi}\nu\epsilon_{abc}F_{bc} \ , \tag{5.23}$$

which agrees with eq. (4.8). When we now compute the conductivity σ_{xy} we get

$$\sigma_{xy} = \nu\frac{q^2}{h} \ , \tag{5.24}$$

and we see that the momentum space topology we uncovered is responsible for its quantization in units of q^2/h. This field theoretic derivation from [4] contains the physics first discovered in the famous TKKN paper by Thouless et al. [5]. When considering lattice domain wall fermions one finds that by adjusting the ratio of the mass the the Wilson coupling one can make ν jump by integer quantities, just as in the original Integer Quantum Hall Effect.

Table 1. *The periodic table of topological insulators and superconductors in d spatial dimensions, assuming non-interacting fermions. On the far right are the names of the ten different topological classes. In the next three columns, matrices $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{C}$ are symmetry matrices using the condensed matter notation (!). To translate to the conventions of relativistic quantum field theory: the matrix $\mathcal{T} = T$, where T the time reversal matrix; $\mathcal{P} = C(\gamma^0)^T$, where C is the charge conjugation matrix, and $\mathcal{C} \equiv \mathcal{TP}$. The + symbols mean that the matrix is symmetric, the − means that it is antisymmetric, and the × symbol means that it is not a symmetry of the theory. On the right side of the table there is a blank if that particular class does not exist in d spatial dimensions, while the $\mathbb{Z}$ and $\mathbb{Z}_2$ entries describe the topological invariant – with $\mathbb{Z}$ there can be any integer number of massless edge states, while for $\mathbb{Z}_2$, edge states can become gapped pairwise.*

	$\mathcal{T}$	$\mathcal{P}$	$\mathcal{C}$	d= 1	2	3	4
A	×	×	×		$\mathbb{Z}$		$\mathbb{Z}$
AIII	×	×	+	$\mathbb{Z}$		$\mathbb{Z}$	
AII	−	×	×		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
DIII	−	+	+	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
D	×	+	×	$\mathbb{Z}_2$	$\mathbb{Z}$		
BDI	+	+	+	$\mathbb{Z}$			
AI	+	×	×				$\mathbb{Z}$
CI	+	−	+			$\mathbb{Z}$	
C	×	−	×		$\mathbb{Z}$		$\mathbb{Z}_2$
CII	−	−	+	$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$

5.4 Other dimensions, other symmetries

We could have deduced the existence of the chiral edge states in the above example by simply computing $j^a_{\rm CS}$ and realizing that its divergence at a boundary meant there had to be chiral fermions at the boundary to receive the charge. But what about theories in other dimensions, or with other sorts of fermions? We have seen that domain wall fermions can exist in any dimension, but chiral anomalies only exist at odd spatial dimension boundaries, and Chern Simons terms only exist in an even spatial dimension bulk. Nevertheless, the edge states can be understood in terms of topology in more exotic cases, and the possiblities have been classified by condensed matter physicists (for free fermions). The topological class depends on whether or not the theory possesses time reversal symmetry $\mathcal{T}$ and/or charge conjugation symmetry $\mathcal{C}$, or whether it has neither individually but does have $\mathcal{CT}$ symmetry. It also depends on whether the time reversal and charge conjugation matrices T and C are symmetric or antisymmetric. Each of these cases constrains the fermion spectrum in a different way. For example, consider a Dirac fermion with charge conjugation symmetry with antisymmetric unitary matrix C which satisfies $C^\dagger \not{D} C = -\not{D}^*$. Then if $\not{D}\phi_n = i\lambda_n\phi_n$, it follows that $\not{D}C\phi_n^* = i\lambda_n C\phi_n^*$ and that $C\phi_n^*$ is orthogonal to ϕ_n. So all eigenvalues are at least doubly degenerate in this theory. The classification is summarized in the periodic table of topological insulators and superconductors, shown in Table 1. Listed are the topological invariants; some are $\mathbb{Z}$ and can take any integer value, such as the winding number ν discussed in this section (a theory in the D class in $d = 2$ spatial dimensions). Others show $\mathbb{Z}_2$, where the domain wall fermions can annihilate pairwise[6]. A reformulation of these classes in a unified framework for relativistic quantum field theory is the subject of references [6, 7].

6 Lattice domain wall fermions

There are many ways that chiral symmetry plays an important role in the Standard Model. One way is that it ensures that fermion masses are multiplicatively renormalized, since they break chiral symmetry, instead of additively. That is why you don't hear people fretting about a fine tuning problem for the quark and lepton masses as they do for the Higgs mass. We also understand relative lightness and properties of the pions as being due to their being approximate Goldstone bosons arising when the quark condensate in QCD spontaneously breaks chiral symmetries of the light quarks. And then there is the fact that the Standard Model is a chiral gauge theory – left- and right-handed fermions do not have the same gauge interactions under $SU(2) \times U(1)$. Therefore it would seem important to be able to simulate quantum field theories on the lattice with exact or approximate chiral symmetry. That is actually difficult to do. Luckily, it turns out the physics we have been discussing can deal with global chiral symmetries on the lattice; they are promising for constructing lattice chiral gauge theories, but that remains an unsolved problem.

6.1 Doubling of a chiral fermion

Let's reconsider the theory of a single gauged right-handed fermion. We now know that the U(1) gauge current will have an anomalous divergence, which means that the theory is not gauge invariant! Such theories are known to be sick, so it should not be possible to give them a definition on the lattice. Let's see what happens if we try.

[6]Note that condensed matter physicists use $\mathcal{P}$ for particle-hole symmetry, related to $\mathcal{C}$ in QFT; and they use $\mathcal{C}$ for what they call "chiral symmetry" for what is called $\mathcal{CT}$ in QFT. Confusing!

To keep things simple, I will first consider a latticized version of the Hamiltonian for the free theory of a single RH Weyl fermion in 1+1 dimensions, which only involves discretizing space, not time. The continuum Hamiltonian is simply

$$H = -i\psi^\dagger \partial_x \psi, \tag{6.1}$$

where ψ is a 1-component fermion field, with naive discretization

$$H = -i\frac{1}{2a}\sum_n c_n^\dagger \left(c_{n+1} - c_{n-1}\right) \tag{6.2}$$

where a is the lattice spacing, and the $c_n, c_n^\dagger$ are fermionic ladder operators at site n:

$$\{c_m, c_n\} = 0 \ , \qquad \{c_m, c_n^\dagger\} = \delta_{mn} \ . \tag{6.3}$$

This theory has an exact $U(1)$ symmetry, which is fermion number:

$$Q = \sum_n c_n^\dagger c_n \ , \qquad [Q, H] = 0 \ . \tag{6.4}$$

This is the symmetry we can gauge. The single-particle eigenstates of H are

$$|p\rangle = \sum_n e^{iapn} c_n^\dagger |0\rangle \tag{6.5}$$

with energy eigenvalue

$$H|p\rangle = E_p|p\rangle \ , \qquad E_p = \frac{\sin ap}{a} \ , \qquad -\frac{\pi}{a} \le p \le \frac{\pi}{a} \ . \tag{6.6}$$

Note from the construction of the state $|p\rangle$ that shifting $p \to p + 2\pi a$ gives back the same state, so p-space is a circle and so taking the above range for p (the Brillouin zone) accounts for all states.

What is the continuum limit of this theory? Naively, the continuum limit $ap \to 0$ gives $E_p = p$, the desired continuum result corresponding to a single right-mover, shown in Fig. 1. However, if we rewrite $p = -\pi + k$, then the $ak \to 0$ limit gives $E_k = -k$, a left-mover! We see that the continuum seems to describe a single massless Dirac fermion in the continuum, with both right and left modes, not a single right-mover. That is because the dispersion relation $E_p = \sin ap/a$ crosses the p-axis in two places, $p = 0$ and $p = \pm\pi$, so there will always be two low energy modes, even as $a \to 0$. Furthermore, the exact $U(1)$ symmetry of the lattice is just fermion number, so if we gauge it, the continuum theory looks like QED in $d = 1 + 1$ with a massless charged Dirac fermion, a sensible vector-like theory with a conserved gauge current, unlike the chiral gauge theory we thought we were latticizing.

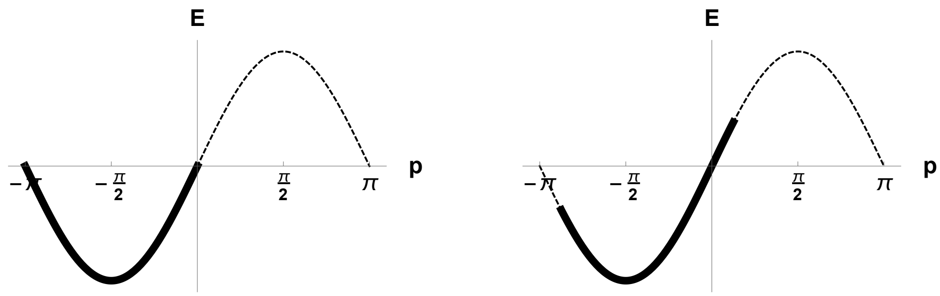

Figure 6. *The ground state of a lattice Dirac fermion (left) and how it has evolved after application of an electric field in the x direction (right). The solid line denotes occupied 1-particle states, and the dashed line vacant states. In the continuum limit, it will appear as if an anomaly has violated axial charge, giving rise to right moving particles and left moving anti-particles.*

Can we add a local interaction that will gap the spectrum at $p = \pm\pi/a$, to get rid of the continuum left-mover? Obviously not: the function E_p must be a continuous function of p (otherwise we will see nonlocal physics in coordinate space) and therefore must be periodic; a periodic function of p cannot cross the p-axis an odd number of times.

And what about the anomalous $U(1)_A$ global symmetry in $d = 1 + 1$ QED? How does the lattice model realize the anomaly? The answer is that the lattice theory does not have a second $U(1)$ symmetry that we can identify with $U(1)_A$ in the continuum...that would require rotating states with $p \sim 0$ with the opposite phase from states near $p \sim \pm\pi/a$, which is not a symmetry of H when I gauge theory (or include any interactions that allow the fermion's momentum to change). Let's look at our heuristic anomaly argument applied to the lattice theory. When we fill all of the negative energy states of H we get the situation pictured on the left in Fig. 6. Now consider what happens when we turn on an electric field for some time in the x direction: all states will move to the right (increasing p up to periodicity) and we end up with the state shown on the right in that figure. In the continuum theory that corresponds to a state that still has no net fermion number, but would have a nonzero axial charge in the continuum limit. The lattice correctly reproduces the axial anomaly by not having a conserved axial charge in the first place. You can see from the heuristic anomaly argument that the problem arises because a finite lattice has a finite number of degrees of freedom, and so there is no Hilbert Hotel to allowing charge to enter and leave the vacuum in each chiral mode independently.

Thinking about anomalies is a good way to understand the doubling problem.

Consider the Fujikawa derivation of the anomaly. Fujikawa showed us that massless vector-like theory in the continuum which has an exact classical chiral symmetry of the action has an anomaly arising from defining the measure. On the lattice though, we have a finite number of degrees of freedom and the Jacobian of the measure is well defined and respects the axial symmetry: when multiplying ψ by $\exp(i\alpha\gamma_\chi)$ and equal number of modes get multiplied by $\exp(i\alpha)$ and $\exp(-i\alpha)$ since γ_χ is traceless, and so the Jacobian equals 1. Therefore if we hope to obtain our continuum theory from the lattice, it must be that the lattice action explicitly violates the chiral symmetry.

The problem is made very precise in the Nielsen-Nionomiya theorem which states succinctly what we cannot do on the lattice. It says that a fermion action in $2k$ Euclidean spacetime dimensions

$$S = \int_{\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^4} \, \overline{\psi}_{-\mathbf{p}} \tilde{D}(\mathbf{p}) \psi(\mathbf{p}) \tag{6.7}$$

cannot have the operator $\tilde{D}$ satisfy all four of the following conditions simultaneously:

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ;
2. $\tilde{D}(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$;
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$;
4. $\{\gamma_\chi, \tilde{D}(\mathbf{p})\} = 0$.

The first condition is required for locality of the Fourier transform of $\tilde{D}(\mathbf{p})$ in coordinate space. The next two state that we want a single flavor of conventional Dirac fermion in the continuum limit. The last item is the statement that we cannot have a chiral symmetry. One can try keeping (4) and eliminating one or more of the other conditions; for example, the SLAC derivative took $\tilde{D}(\mathbf{p}) = \gamma_\mu p_\mu$ within the Brillouin zone (BZ), which violates the first condition — if taken to be periodic, it is discontinuous at the edge of the BZ. This causes problems — for example, the QED Ward identity states that the photon vertex Γ_μ is proportional to $\partial\tilde{D}(\mathbf{p})/\partial p_\mu$, which is infinite at the BZ boundary. Naive fermions satisfy all the conditions except (3): there $\tilde{D}(\mathbf{p})$ vanishes at the 2^4 corners of the BZ, and so we have 2^4 flavors of Dirac fermions in the continuum. Staggered fermions are somewhat less redundant, producing four flavors in the continuum for each lattice field. The discussion in any even spacetime dimension is analogous.

For many years it looked like the only options were to violate (2) and have the wrong number of flavors, or else violate (4) and break chiral symmetry explicitly. Wilson fermions are a solution of the latter form, where one adds a dimension 5 operator to the theory, resembling the continuum theory:

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + M + \frac{r}{2\Lambda}\nabla^2\psi \ , \tag{6.8}$$

where r is a dimensionless coupling and Λ is the UV cutoff, corresponding to the inverse lattice spacing $1/a$. This gives us

$$\tilde{D}(p) = \sum_\mu \left[\frac{i}{a}\sin[ap_\mu]\gamma_\mu + \frac{r}{a}(\cos ap_\mu - 1)\right] + M \tag{6.9}$$

which solves the doubling problem by vanishing only at $p_\mu = 0$ (satisfying (3)), but which violates chiral symmetry even with $M = 0$ since $\bar{\psi}\nabla^2\psi$ has the same Lorentz structure as a mass term. Since setting $M = 0$ does not enhance the symmetry of the theory, M can get additively renormalized, and the natural scale for the fermion mass will be t$O(1/a)$. To obtain a light fermion requires fine-tuning M against r.

An interesting question to ask is: is it possible to violate (4) in a minimal way, so that anomalies are reproduced, but in all other ways one sees good chiral symmetry, protecting against additive mass renormalization, for example. The answer is yes, and an implicit answer for how chiral symmetry has to be modified was provided by Ginsparg and Wilson in the 1980s [8]. They said that the Nielsen-Ninomiya theorem was realized by preserving properties (1-3), while having

$$\{\gamma_\chi, \tilde{D}\} = a\tilde{D}\gamma_\chi\tilde{D} \ . \tag{6.10}$$

I won't derive it here (see, for example, [9]) but I can motivate it as following. First of all, consider an insertion of $\partial_\mu j^\mu_A$ at the vertex of the anomaly diagram in Fig. **??**. If eq. (6.10) holds, then if we look at an insertion of $\partial_\mu j^\mu_A$ in some diagram, we can replace that operaotr by an insertion of $a\bar{\psi}\tilde{D}\gamma_\chi\tilde{D}\psi$. Since $\tilde{D}$ is the inverse propagator the factors of $\tilde{D}$ will cancel the propagators coming out of the vertex; since the Fourier transform of 1 in momentum space is $\delta^d(x)$ in position space, this means that only singular UV contribiutions to the diagram are affected, and long range effects, such as additive renormalization to the fermion mass by radiative effects of another particle, should not be affected. The anomaly, however, arises exactly because of the singularities when currents sit on top of each other. The short distance nature of eq. (6.10) is perhaps better seen by multiplying on both sides by $\tilde{D}^{-1}$ and then Fourier transforming to coordinate space:

$$\{\gamma_\chi, D^{-1}\}_{xy} = a\gamma_\chi\,\delta^d(x-y) \tag{6.11}$$

again showing that chiral symmetry violation with Ginsparg-Wilson fermions is strangely mild. In fact, one can show that Ginsparg-Wilson fermions respect an exact U(1) symmetry on the lattice (a not exactly local transformation) which serves to protect fermion masses from additive renormalization, while still allowing the anomaly (and index theorem) to be correctly reproduced [10].

The only problem with eq. (6.10) is that for a decade no one had a solution for a fermion operator $\tilde{D}$ which satisfied it.

The path to a solution started with the observation that one could create a lattice theory of domain wall fermions, with an RH zeromode on the domain wall, and a LH zeromode on the anti-domain wall. Then one adds gauge fields which are independent of the extra dimension. If one could ignore the heavy fermions in the bulk, then the low energy effective theory would be of a Dirac fermion with exponentially small mass, proportional to $m_0 \exp(-2m_0 L)$, where m_0 is the domain wall height and L the separation between the two walls. This becomes an exactly massless fermion in the $L \to \infty$ limit. Interactions cannot gap the fermion (radiately generate a mass) because of the physical separation of the modes in the extra dimension, so the theory would have an exact chiral symmetry and multiplicative mass renormalization. However, we have seen that a Chern-Simons operator is generated when integrating out the bulk fermions, and so chiral symmetry is not exact, but broken exactly in the way the anomaly is supposed to break it. This picture was put on firm ground when Neuberger, based on earlier work with Narayanan, computed the effective fermion operator $\tilde{D}$ of the domain wall fermion in the limit $L \to \infty$,. the so-called "Overlap Operator", and showed that it provided the first explicit solution to the Ginsparg-Wilson equation eq. (6.10) [11].

It seems we should be happy that the lattice was "smart enough" to not give us a lattice theory with an exact chiral symmetry that we could gauge, when in the continuum limit such a theory has a gauge anomaly and makes no sense. The problem is the following: if a chiral symmetry is broken by anomalies in the continuum, it must be explicitly broken by the lattice action, since the path integral measure for a finite Hilbert space will not exhibit the anomaly Fujikawa discovered in the continuum. If it is explicitly broken on the lattice in order to correctly reproduce anomalies, it will typically not exist at all and so, for example, fermion masses will be additively renormalized. Furthermore, simulating chiral gauge theories is problematic. In the continuum if I have two RH Weyl fermions, each of which has an anomalous fermion number symmetry, I may be able to gauge a linear combination for which the anomalies cancel (for example, our 3-4-5 U(1) gauge theory in 1+1 dimensions). However, on the lattice, each of these symmetries is broken explicitly in the action, so for free fermions at least, it seems impossible to get the symmetry breaking to cancel.

Again, I want to emphasize the idea that the roadblock to developing a lattice theory with chirality is the existence of anomalies in the continuum. Any symmetry that is exact on the lattice will be exact in the continuum limit, while any symmetry anomalous in the continuum limit must be broken explicitly on the lattice. In principle though this should not be a fatal obstacle. After all, in the continuum, chiral symmetry can be broken by an anomaly, while still protecting a fermion from additive mass renormalization, at least in a theory like QED. So the question we should be posing is: can we violate (5) in just the right way to reproduce the anomalies, while preserving all the desired features of chiral symmetry?

It turns out that the answer is "yes" and that the key lies in a very simple model of fermions in the continuum: a free massive Dirac fermion living in odd spacetime dimensions which has a discontinuity in the value of its mass — either the fermion lives on a space with a boundary from which it cannot escape, or else it lives in a space where its mass changes sign somewhere. In discussing this, we will see that all of the things we have been learning about anomalies in different dimensions will come together.

6.2 Domain wall fermions on the lattice

The next step is to transcribe this theory onto the lattice. If you replace continuum derivatives with the usual lattice operator $D \to \frac{1}{2}(\nabla^* + \nabla)$ (where ∇ and ∇^* are the forward and backward lattice difference operators respectively) then one discovers...doublers! Not only are the chiral modes doubled in the $2k$ dimensions along the domain wall, but there are two solutions for the transverse wave function of the zero mode, $b_0(s)$, one of which alternates sign with every step in the s direction and which is a LH mode. So this ends up giving us a theory of naive fermions on the lattice, only in a much more complicated and expensive way!

However, when we add a Wilson term $\frac{r}{2}\bar{\psi}\sum_i \nabla_i^* \nabla_i \psi$ for each of the dimensions $i = 1, \ldots, 4$, things get interesting. You can think of these as mass terms which are independent of s but which are dependent on the wave number k of the mode, vanishing for long wavelength. What happens of we add a k-dependent spatially constant mass $\Delta m(k)$ to the step function mass $m(s) = m\epsilon(s)$? The solution for $b_0(s)$ in eq. (5.5) for an infinite extra dimension becomes

$$b_0 = N e^{-\int^L [m(s')+\Delta m(k)]ds'} , \tag{6.12}$$

which is a normalizable zeromode solution — albeit, distorted in shape — so long as $|\Delta m(k)| < m$. However, for $|\Delta m(k)| > m$, the chiral mode vanishes. What happens to it? It becomes more and more extended in the extra dimension until it ceases to be normalizable. What is going on is easier to grasp for a finite extra dimension: as $|\Delta m(k)|$ increases with increasing k, eventually the b_0 zeromode solution extends to

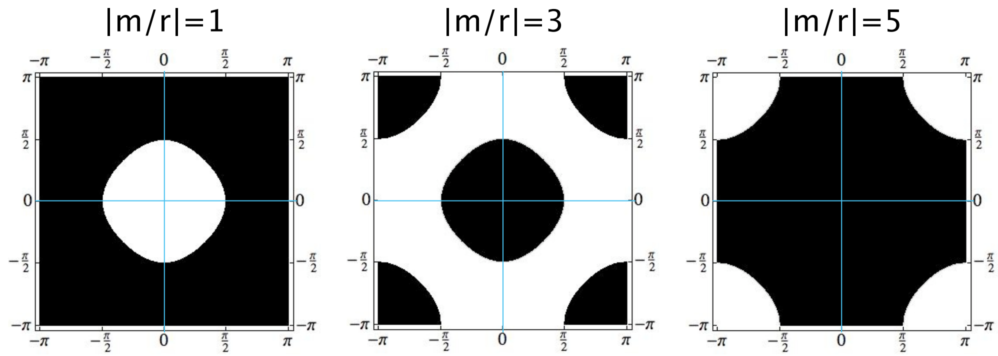

Figure 7. *Domain wall fermions in $d = 2$ on the lattice: dispersion relation plotted in the Brillouin zone. Chiral modes exist in white regions only. For $0 < |m/r| < 2$ there exists a single RH mode centered at $(k_1, k_2) = (0, 0)$. for $2 < |m/r| < 4$ there exist two LH modes centered at $(k_1, k_2) = (\pi, 0)$ and $(k_1, k_2) = (0, \pi)$; for $4 < |m/r| < 6$ there exists a single RH mode centered at $(k_1, k_2) = (\pi, \pi)$. For $|m/r| > 6$ there are no chiral mode solutions.*

the opposing boundary of the extra dimension, when $|\Delta m(k)| \sim (m - 1/L)$. At that point it can pair up with the LH mode and become heavy.

So the idea is: add a Wilson term, with strength such that the doublers at the corners of the Brillouin zone have $|\Delta m(k)|$ too large to support a zeromode solution. Under separation of variables, one looks for zeromode solutions with $\psi(x, s) = e^{ipx}\phi_\pm(s)\psi_\pm$ with $\gamma_\chi \psi_\pm = \pm\psi$. One then finds (for $r = 1$)

$$\not{p}_4 \psi_\pm = 0 \; , \qquad -\phi_\pm(s \mp 1) + (m_{\text{eff}}(s) + 1)\phi_\pm(s) = 0 \quad , \tag{6.13}$$

where

$$m_{\text{eff}}(s) = m\epsilon(s) + \sum_\mu (1 - \cos p_\mu) \equiv m\epsilon(s) + F(p) \; . \tag{6.14}$$

Solutions of the form $\phi_\pm(s) = z_\pm^s$ are found with

$$z_\pm = (1 + m_{\text{eff}}(s))^{\mp 1} = (1 + m\epsilon(s) + F(p))^{\mp 1} \; ; \tag{6.15}$$

they are normalizable if $|z|^{\epsilon(s)} < 1$. Solutions are found for ψ_+ only, and then provided that m is in the range $F(p) < m < F(p) + 2$. (For $r \neq 1$, this region is found by replacing $m \to m/r$.) However, even though the solution is only found for ψ_+, the chirality of the solutions will alternate with corners of the Brillouin zone, just as we found for naive fermions. The picture for the spectrum in 2d is shown in Fig. 7. It was first shown in [12] that doublers could be eliminated for domain wall fermions on the lattice; the rich spectrum in Fig. 7 was worked out in [13], where for 4d they found the number of zeromode solutions to be the Pascal numbers $(1, 4, 6, 4, 1)$ with alternating chirality, the critical values for $|m/r|$ being $0, 2, \ldots, 10$. One implication of their work is that the Chern Simons currents must also change discontinuously on the lattice at these critical values of $|m|/r$; indeed that is the case: one can reproduce the calculation we did for the coefficient of the Chern-Simons operator in § 5.3 using the propagator for Wilson fermions, assuming smooth gauge fields, and one finds that the winding number ν that one calculates jumps discontinuously at the magic values of m/r (which is where the bulk goes gapless, allowing a topological phase transition), and that the jumps in ν are exactly what one needs to explain the change in number and chirality of domain wall fermions [14].

At finite lattice spacing the phase diagram is expected to look something like in Fig. 8 where I have plotted m versus g^2, the strong coupling constant. On this diagram, $g^2 \to 0$ is the continuum limit. Domain wall fermions do not require fine tuning so long as the mass is in one of the distinct topological phases marked by an "X", which yield $\{1, 4, 6, 4, 1\}$ chiral flavors from left to right. The shaded region is a phase called the Aoki phase [15]; it is presently unclear whether the phase extends to the continuum limit (left side of Fig. 8) or not (right side) [16]. In either case, the black arrow indicates how for Wilson fermions on tunes the mass from the right to the boundary of the Aoki phase to obtain massless pions and chiral symmetry; if the Aoki phase extends down to $g^2 = 0$ than the Wilson program will work in the continuum limit, but not if the RH side of Fig. 8 pertains. See [17, 18] for a sophisticated discussion of the physics behind this diagram.

6.3 Chiral gauge theories: the challenge

Unfortunately, while the Ginsparg-Wilson equation, domain wall fermions, and the overlap operator now allow us to simulate vector-like theories with global chiral symmetry, they still do not tell us how to simulate chiral gauge theories. That is because the domain wall and anti-domain wall host zeromodes with identical gauge charges but opposite chirality: eg, they give one a vector-like theory. In order to simulate a chiral gauge theory, a popular idea is that one put the theory of interest on one domain wall, the mirror theory on the other, and then somehow render the mirror fermions invisible. One suggestion is to not let the gauge fields go there, breaking gauge invariance, and only restoring gauge invariance by fine tuning as one goes to the continuum limit [19]. Another is to try to construct chiral gauge theories using the overlap operator, which was succeeded for $U(1)$ but not nonabelian gauge theories [20]. Another was to give the mirror fermions infinitely soft form factors so that gauge fields couldn't couple to them [21, 22]. Finally, it has been proposed to gap the mirror fermions via strong inter-

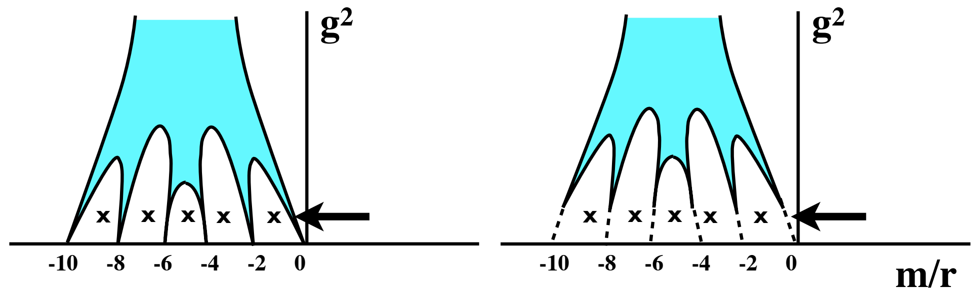

Figure 8. *A sketch of the possible phase structure of QCD with Wilson fermions where the shaded region is the Aoki phase — pictured extending to the continuum limit (left) or not (right). When using Wilson fermions one attempts to tune the fermion mass to the phase boundary (arrow) to obtain massless pions; this is only possible in the continuum limit of the picture on the left is correct. For domain wall fermions chiral symmetry results at infinite L when one simulates in any of the regions marked with an "X". There are six "fingers" in this picture instead of five due to the discretization of the fifth dimension.*

actions (given that one cannot give them a gauge invariant mass to do so) [23]. I will obliquely comment on this latter approach in the next section of these lectures; there are recent preliminary indications that this may be possible to do [24]. However, in summary I will say that successfully constructing a nonperturbative version of a chiral gauge theory in four spacetime dimensions has not been convincingly done yet.

7 Majorana edge states

If one wants to make a quantum computer two types of errors one must guard against are classical errors (where a bit flips from $|0\rangle$ to $|1\rangle$) and phase errors (where the state with the n^{th} bit equal to $|1\rangle$ picks up a phase relative to the state where the n^{th} bit is equal to $|0\rangle$). If the qubit was constructed from electrons with $|0\rangle$ and $|1\rangle$ corresponding to an occupied or unoccupied state at a given site on a lattice, then a classical flip would require that an electron hops onto or off of that site from/to another site. This coul be guarded against by having the sites be well separated by a gapped region which the electron couldn't penetrate. However, a phase error would be hard to protect against if, for example, if electrons at different sites had different energies. If that was modeled by an interaction between the electrons and a background field, such as by $\delta H = \phi_n a_n^\dagger a_n$, where a_n is the electron annihilation operator at site n, then a state where site n is occupied picks up a phase $e^{-i\phi_n t}$ relative to the state where it is unoccupied.

Kitaev [25] proposed that this might be remedied with Majorana fermions. The idea is that at each site one writes a_n in terms of two real operators:

$$a_j = \frac{c_{2j-1} + ic_{2j}}{2} \ , \qquad a_j^\dagger = \frac{c_{2j-1} - ic_{2j}}{2} \ . \tag{7.1}$$

Then the anticommutation relations $\{a_j, a_k\} = 0$, $\{a_j, a_k^\dagger\} = \delta_{jk}$ imply

$$\{c_m, c_n\} = 2\delta_{mn} \ . \tag{7.2}$$

This looks like a Clifford algebra, familiar from the study of Dirac gamma matrices. Note that with the operators a and $a^\dagger$ we can construct a complete set of operators on a 2-state system. Their matrix form can be taken to be

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \qquad a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \tag{7.3}$$

and $\{1, a, a^\dagger, a^\dagger a\}$ form a complete set of 2×2 matrices. However, a single Majorana operator c on a 2-state system can be taken to be σ_1, and then we cannot construct from c anything more than the unit operator and c itself, which does not for a complete operator basis for the 2-state system. Therefore a single Majorana fermion cannot adequately describe the Hilbert space; we need them in pairs.

Now consider a phase error created by the $a_j^\dagger a_j = (1 + 2ic_{2j-1}c_{2j})/4$ operator. The identity part is uninteresting as it doesn't care about occupation number. What is intriguing about the $c_{2j-1}c_{2j}$ term is that if one could ensure that the Majorana operator c_{2j-1} acted on a site physically far away from the site where c_{2j} acts, then such phase errors would be unlikely, since they could only arise from a highly nonlocal operator. Kitaev went on to show that one can create such a situation with Majorana domain wall fermions, where ungapped Majorana modes are separated physically and stuck on different domain walls in 1+1 dimensions (eg, opposite ends of a wire) [7].

Kitaev's model took the form of the Hamiltonian

$$\begin{aligned} H &= \sum_j \left[-w(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) - \mu(a_j^\dagger a_j - \tfrac{1}{2}) + \Delta(a_j a_{j+1} + a_{j+1}^\dagger a_j^\dagger) \right] \\ &= \frac{i}{2} \left[-\mu c_{2j-1}c_{2j} + (\Delta + w)c_{2j}c_{2j+1} + (\Delta - w)c_{2j-1}c_{2j+2} \right] \ . \end{aligned} \tag{7.4}$$

If we set $\Delta = w$ we can rewrite this as

$$H = \frac{i}{2} \sum_j \left(uc_{2j-1}c_{2j} + vc_{2j}c_{2j+1} \right) \ . \tag{7.5}$$

[7] Majorana edge states had been proposed earlier in the context of simulating the gluinos in $N = 1$ super Yang-Mills theory on the lattice [26, 27].

If we massage this further by defining

$$c_{2j} = \eta_{2j}^{(1)} + \eta_{2j}^{(2)} \ , \qquad c_{2j-1} = \eta_{2j-1}^{(1)} - \eta_{2j-1}^{(2)} \ , \tag{7.6}$$

and one takes the continuum limit and considers the fields $\eta^{(1,2)}(x)$, one arrives at the Hamiltonian

$$H = \frac{i}{2}\left[\eta^{(1)}\partial_x\eta^{(1)} - \eta^{(2)}\partial_x\eta^{(2)} + 2m\eta^{(2)}\eta^{(1)}\right] \ . \tag{7.7}$$

Finally, we can derive this Hamiltonian from a simple field theory of a real, 2-component Majorana spinor in $1+1$ dimensions,

$$\mathcal{L} = \tfrac{1}{2}\psi^T C\left(i\partial\!\!\!/ - m\right)\psi \ , \qquad \psi(x) = \begin{pmatrix}\eta^{(1)}(x)\\ \eta^{(2)}(x)\end{pmatrix} \ , \tag{7.8}$$

with

$$\gamma^0 = C = \sigma_2 \ , \quad \gamma^1 = -i\sigma_1 \ , \quad \gamma_\chi = \sigma_3 \ . \tag{7.9}$$

Now consider what happens if we have a domain wall, $m(x) = m\epsilon(x)$. There are two solutions to $(i\partial\!\!\!/ - m)\psi = 0$, namely

$$\psi_\pm = e^{\pm m|x|}\begin{pmatrix}1\\ \pm 1\end{pmatrix} \ , \tag{7.10}$$

and as usual we toss out ψ_+ because it is not normalizable, and retain a single zeromode bound to the mass defect. The low energy theory therefore is a $0+1$ dimension theory (eg, quantum mechanics) of

$$\psi_-(x,t) = \frac{\eta(t)}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} e^{-m|x|} \ , \tag{7.11}$$

and if we insert this into our Lagrange density and integrate over x we get the simple Lagrangian for a single real degree of freedom

$$L = \frac{1}{2}\eta i\partial_t\eta \ . \tag{7.12}$$

Quantizing this gives $\{\eta,\eta\} = 1$.

As I mentioned earlier, this single Majorana mode cannot be used to build a sensible Hilbert space. However, if we have a finite system, for example with periodic BC in the x direction, the ψ_+ solution gives us zeromode at the anti-domain wall, and the two Majoranas together give us a Hilbert space. This is exactly the situation we were hoping for, where electrons got split, with half of an electron on one edge, half on the other, with a gapped region between. So the phase error would have to be a nonlocal operator encompassing both domain walls, which hopefully is very unlikely to occur.

The Kitaev model is interesting for pointing the way to solve a different problem: defining chiral gauge theories. Consider several copies (flavors) of the model, the low energy theory is

$$L = \frac{1}{2}\eta_i i\partial_t \eta_i \ . \tag{7.13}$$

with $\{\eta_i, \eta_j\} = \delta_{ij}$ after quantization. When I have an even number of flavors I can define a Hilbert space locally, by using the eta_i on a single domain wall pairwise to define fermionic ladder operators. An immediate question is: is there any symmetry that keeps them from getting gapped? If there is, it would be an analog of chiral gauge symmetry which prohibits gapping the mirror version of the Standard Model located on the other domain wall from where we live. For example, when we have two flavors, look what happens if we add the following hermitian operator to the $1 + 1$ dimension Lagrangian:

$$\delta\mathcal{L} = \mu\psi_1^T C\gamma_5\psi_2 = i\mu\psi_1\sigma_1\psi_2 \ , \qquad \mu \in \Re \ . \tag{7.14}$$

If we plug in our two zeromode solutions and integrate over x, our modified effective theory is

$$L + \delta L = \frac{1}{2}\eta_i i\partial_t \eta_i - i\mu\eta_1\eta_2 \ , \tag{7.15}$$

which shows that we have gapped the system.

The answer is yes: it is time reversal invariance forbids such a term. Under time reversal, $\psi(x, t) \to \sigma_1\psi(x, -t)$, and since it is anti-unitary, it flips the sign of the i and hence $\delta\mathcal{L}$. So time reversal keeps is from simply adding quadratic terms to the Lagrangian to gap the fermions.... just the way the chiral gauge theory does in the more interesting 3+1 dimension example.

So an interesting question is: can we add time-reversal invariant four-fermion interactions which gap the edge states on a single domain wall? The answer was provided by Fidkowski and Kitaev: the answer is yes, but one can only gap multiples of eight flavors at a time [28]. A relatively simple explanation of this is given in Appendix B of Ref. [7]. The idea is that $2N$ zeromode operators η_i can be represented by the $2N$ gamma matrices of $SO(2N)$, since they are a fine representation of the Clifford algebra. From these one can define N complex fermionic ladder operators, eg, $a_i = \eta_i + i\eta_{i+N}$, $a_i^\dagger = \eta_i - i\eta_{i+N}$. These define a 2^N dimensional Hilbert space, which transforms as a spinor under $SO(2N)$, and the η_i act on these states the way the γ matrices act on the 2^N dimensional spinor of $SO(2N)$. So a four-fermion interaction term is represented as a 2^N dimensional matrix constructed as a sum of products of four gamma matrices. The question of whether the edge states can be completely gapped then becomes a question about whether such a matrix can have a unique lowest eigenvalue. It turns out that this requires having a C matrix and a $T = C\gamma_5$ matrix with certain symmetry properties... and one finds that these properties are only found in $SO(2N)$ when $2N = 0 \mod (8)$.

There has been a lot of recent research on whether one can use this sort of mechanism to realize the old Eichten-Preskill idea to gap mirror fermions in a vector-like realization of a chiral gauge theory, giving rise to a sensible chiral gauge theory, but in my mind there has not yet been a convincing argument that candidate theories actually possess the require phase at strong coupling.

References

[1] M.Z. Hasan and C.L. Kane, *Colloquium: topological insulators*, *Reviews of modern physics* **82** (2010) 3045.

[2] X.-L. Qi and S.-C. Zhang, *Topological insulators and superconductors*, *Reviews of Modern Physics* **83** (2011) 1057.

[3] J. Callan, Curtis G. and J.A. Harvey, *Anomalies and Fermion Zero Modes on Strings and Domain Walls*, *Nucl. Phys.* **B250** (1985) 427.

[4] M.F. Golterman, K. Jansen and D.B. Kaplan, *Chern-simons currents and chiral fermions on the lattice*, *Physics Letters B* **301** (1993) 219.

[5] D.J. Thouless, M. Kohmoto, M.P. Nightingale and M. den Nijs, *Quantized Hall conductance in a two-dimensional periodic potential*, *Physical review letters* **49** (1982) 405.

[6] D.B. Kaplan and S. Sen, *Index theorems, generalized hall currents and topology for gapless defect fermions*, 2112.06954.

[7] D.B. Kaplan and S. Sen, *Generalized hall currents in topological insulators and superconductors*, 2205.05707.

[8] P.H. Ginsparg and K.G. Wilson, *A Remnant of Chiral Symmetry on the Lattice*, *Phys. Rev.* **D25** (1982) 2649.

[9] D.B. Kaplan, *Chiral Symmetry and Lattice Fermions*, in *Les Houches Summer School: Session 93: Modern perspectives in lattice QCD: Quantum field theory and high performance computing*, pp. 223–272, 12, 2009 [0912.2560].

[10] M. Luscher, *Exact chiral symmetry on the lattice and the Ginsparg- Wilson relation*, Phys. Lett. **B428** (1998) 342 [hep-lat/9802011].

[11] H. Neuberger, *Exactly massless quarks on the lattice*, Phys. Lett. **B417** (1998) 141 [hep-lat/9707022].

[12] D.B. Kaplan, *A Method for simulating chiral fermions on the lattice*, Phys. Lett. **B288** (1992) 342 [hep-lat/9206013].

[13] K. Jansen and M. Schmaltz, *Critical momenta of lattice chiral fermions*, Phys. Lett. **B296** (1992) 374 [hep-lat/9209002].

[14] M.F.L. Golterman, K. Jansen and D.B. Kaplan, *Chern-Simons currents and chiral fermions on the lattice*, Phys. Lett. **B301** (1993) 219 [hep-lat/9209003].

[15] S. Aoki, *New Phase Structure for Lattice QCD with Wilson Fermions*, Phys. Rev. **D30** (1984) 2653.

[16] M. Golterman, S.R. Sharpe and J. Singleton, Robert L., *Effective theory for quenched lattice QCD and the Aoki phase*, Phys. Rev. **D71** (2005) 094503 [hep-lat/0501015].

[17] M. Golterman and Y. Shamir, *Overlap-Dirac fermions with a small hopping parameter*, *JHEP* **09** (2000) 006 [hep-lat/0007021].

[18] M. Golterman and Y. Shamir, *Localization in Lattice QCD*, Phys. Rev. **D68** (2003) 074501 [hep-lat/0306002].

[19] M. Golterman and Y. Shamir, *SU(N) chiral gauge theories on the lattice*, Phys. Rev. **D70** (2004) 094506 [hep-lat/0404011].

[20] M. Luscher, *Chiral gauge theories revisited*, *arXiv: hep-th/0102028* (2000) [hep-th/0102028].

[21] D.M. Grabowska and D.B. Kaplan, *Nonperturbative regulator for chiral gauge theories?*, *Physical review letters* **116** (2016) 211602.

[22] D.M. Grabowska and D.B. Kaplan, *Chiral solution to the ginsparg-wilson equation*, *Physical Review D* **94** (2016) 114504.

[23] E. Eichten and J. Preskill, *Chiral gauge theories on the lattice*, *Nucl. Phys.* **268** (1985) 179.

[24] J. Wang and Y.-Z. You, *Symmetric Mass Generation*, 2204.14271.

[25] A.Y. Kitaev, *Unpaired majorana fermions in quantum wires*, *Physics-Uspekhi* **44** (2001) 131.

[26] D.B. Kaplan, *Supersymmetric Yang-Mills Theories (On The Lattice)*, *Talk given at CHIRAL 99, Taipei, September 13-18 1999, http://chiral.phys.ntu.edu.tw/program.html* (1999) .

[27] D.B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543 [hep-lat/0002030].

[28] L. Fidkowski and A. Kitaev, *The effects of interactions on the topological classification of free fermion systems*, *Phys. Rev. B* **81** (2010) 134509 [0904.2197].