Physics 572

Problem set #4 **Due Tuesday 5/24/2016**

---

1. Writing $\Sigma = \sqrt{2}i\phi/f_\pi$ where ϕ is the meson matrix

$$\phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2\eta}{\sqrt{6}} \end{pmatrix} \tag{1}$$

then we get to quadratic order (setting $Q_L = Q_R = Q$)

$$\mathcal{L}_{\text{em}} = c\frac{\alpha}{4\pi}f_\pi^4 \text{Tr}\left[Q^2 + \frac{\sqrt{2}i}{f_\pi}Q[\phi, Q] + \frac{1}{2}\left(\frac{\sqrt{2}i}{f_\pi}\right)^2 Q[\phi,[\phi,Q]] + \dots\right] . \tag{2}$$

This term only contributes to the charged meson masses, so we can simplify the calculation by setting all the meson fields to zero except the $\pi^\pm$ and find

$$\mathcal{L}_{\text{em}} = -cf_\pi^2\frac{\alpha}{2\pi}\pi^+\pi^- . \tag{3}$$

Thus we get

$$m_{\pi^+}^2 - m_{\pi^0}^2 = \frac{c\alpha}{2\pi}f_\pi^2 , \qquad c = \frac{2\pi}{\alpha}\frac{m_{\pi^+}^2 - m_{\pi^0}^2}{f_\pi^2} = 137. \tag{4}$$

This is a large value for c and one might worry that perturbation theory in α is not working (and it is very weird that $c \simeq 1/\alpha$ to high accuracy), but this value looks OK if you use "naive dimensional analysis" to reexpress the operator in terms of the "natural" size:

$$\mathcal{L}_{\text{em}} = \frac{c}{(4\pi)^2}\frac{\alpha}{4\pi}f_\pi^2\Lambda_\chi^2 \text{Tr}\, Q_L\Sigma Q_R\Sigma^\dagger , \qquad \Lambda_\chi \equiv 4\pi f_\pi . \tag{5}$$

With this naive dimensional analysis power counting, where operator coefficients are expected to be $O(1)$, the coefficient of interest is

$$\frac{c}{16\pi^2} = 0.867 . \tag{6}$$

which is gratifyingly close to one.

2. *7 points out of 10.* The ultraquarks are gauged with the following charges under $SU(2)\times U(1)\times U(1)'$:

$$\begin{pmatrix} U \\ D \end{pmatrix}_L = 2_{\frac{1}{6},\frac{1}{6}} , \qquad S_L = 1_{-\frac{1}{3},-\frac{1}{3}} , \qquad \begin{pmatrix} U \\ D \end{pmatrix}_R = 2_{\frac{1}{6},-\frac{1}{6}} , \qquad S_R = 1_{-\frac{1}{3},\frac{1}{3}} . \tag{7}$$

and therefore the Σ field transforms under $SU(2)\times U(1)\times U(1)'$ as

$$SU(2): \quad \delta_a\Sigma = iT_L^a\Sigma - i\Sigma T_R^a , \qquad T_L^a = T_R^a = \begin{pmatrix} \frac{\sigma_a}{2} & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

$$U(1): \quad \delta\Sigma = iY_L\Sigma - i\Sigma Y_R , \qquad Y_L = Y_R = \begin{pmatrix} \frac{1}{6} & & \\ & \frac{1}{6} & \\ & & -\frac{1}{3} \end{pmatrix} \equiv Y \tag{9}$$

$$U(1)': \quad \delta\Sigma = iY_L'\Sigma - i\Sigma Y_R' , \qquad Y_L' = -Y_R' = Y . \tag{10}$$

(a) When we replace Σ by the unit matrix $\mathbf{1}$ in the above transformations, we see that the $SU(2)\times U(1)$ transformations of Σ vanish, while the $U(1)'$ transformation does not. Therefore this vacuum preserves $SU(2)\times U(1)$ but breaks $U(1)'$. We expect one Goldstone boson to be eaten by the $U(1)'$ gauge boson, which gets heavy. If we write $\Sigma = \sqrt{2}i\phi/F_\pi$ with ϕ as in eq. (1), and work to linear order in ϕ, then the $U(1)'$ transformation in eq. (10) is

$$\delta\phi = \sqrt{2}F_\pi Y \qquad \text{or:} \quad \delta\eta = \frac{F_\pi}{\sqrt{3}} \ . \tag{11}$$

The Goldstone boson that shifts under the broken symmetry is the one that can be gauged away and therefore the η is the eaten one.

(b) The analogue of the electromagnetic contribution to the mesons for this ultracolor theory is

$$\mathcal{L}_{\text{gauge}} = cF_\pi^4\left[\frac{\alpha_2}{4\pi}\text{Tr}\, T^a\Sigma T^a\Sigma^\dagger + \frac{(\alpha_1-\alpha_1')}{4\pi}\text{Tr}\, Y\Sigma Y\Sigma^\dagger\right] \tag{12}$$

where a is summed over $a = 1,2,3$, with $\alpha_2 = g_2^2/4\pi$, $\alpha_1 = g_1^2/4\pi$, $\alpha_1' = (g_1')^2/4\pi$. So long as the ultracolor group is $SU(3)$ just like QCD, we can use the result for c from problem 1, $\frac{c}{16\pi^2} = 0.867$.

(c) We can use the analogue of eq. (2) to compute all the meson masses, substituting the couplings and gauge generators shown in eq. (12), with the result that at quadratic order

$$\mathcal{L}_{\text{gauge}} = -\frac{c}{8\pi}F_\pi^2\left[(8\alpha_2)\left(\pi^+\pi^- + \frac{1}{2}(\pi^0)^2\right) + (3\alpha_2+\alpha_1-\alpha_1')\left(K^+K^- + \bar{K}^0K^0\right)\right] \tag{13}$$

so that we find

$$m_\pi^2 = \alpha_2\frac{c}{\pi}F_\pi^2 \ , \qquad m_K^2 = (3\alpha_2+\alpha_1-\alpha_1')\frac{c}{8\pi}F_\pi^2 \ , \qquad m_\eta^2 = 0 \ . \tag{14}$$

The vanishing of the η mass was expected since it is the Goldstone mode that gets eaten by the Higgs mechanism; the pion is an $SU(2)$ triplet with positive mass squared; and the kaons form a complex $SU(2)$ doublet (like the Higgs doublet in the SM) with a mass squared which gets negative contributions from the $U(1)'$ gauge interactions. If we crank up α_1' then the kaons get a negative mass squared, a sign that the vacuum we chose, $\Sigma = \mathbf{1}$ is unstable. The critical value for this instability is

$$\alpha_1' \le \alpha_c \equiv 3\alpha_2 + \alpha_1 \ , \qquad g_{1c}' = \sqrt{3g_2^2+g_1^2} = e\sqrt{\frac{3}{\sin^2\theta_w} + \frac{1}{\cos^2\theta_w}} \tag{15}$$

(d) To investigate what happens at large α_1 we want to look at a more general class of vacua. The obvious one is to consider a vacuum with nonzero vev in one of the kaon (Higgs) directions. Therefore we consider

$$\Sigma = \exp\left[i\begin{pmatrix}0&0&0\\0&0&\theta\\0&\theta&0\end{pmatrix}\right] = \begin{pmatrix}1&0&0\\0&\cos\theta&i\sin\theta\\0&-i\sin\theta&\cos\theta\end{pmatrix} \tag{16}$$

Then

$$\begin{aligned} V(\theta) = -\mathcal{L}_{\text{gauge}} &= \frac{cF_\pi^4}{16\pi}\left[\text{const.} - 4\alpha_2\cos\theta - (\alpha_2+\alpha_1-\alpha_1')\cos^2\theta\right] \\ &= \frac{cF_\pi^4}{16\pi}\left[\text{const.} - 4\alpha_2\cos\theta + (\alpha_1'-\alpha_c+2\alpha_2)\cos^2\theta\right] \\ &= \frac{cF_\pi^4}{16\pi}\left[\text{const.} + 2\alpha_2(\cos\theta-1)^2 + (\alpha_1'-\alpha_c)\cos^2\theta\right] \ , \end{aligned} \tag{17}$$

where α_c is given in eq. (15). For $\alpha_1' \le \alpha_c$ we see that both terms depending on θ are minimized for $\cos\theta = 1$, which is the $SU(2)\times U(1)$ symmetric vacuum $\langle\Sigma\rangle = \mathbf{1}$. For $\alpha_1' > \alpha_c$ the $O(\theta^2)$ term in $V(\theta)$ turns negative and the symmetric vacuum becomes unstable. The minimum condition is

$$\frac{\partial V}{\partial\theta} = 0 = \sin\theta\left(4\alpha_2(\cos\theta - 1) + 2(\alpha_1' - \alpha_c)\cos\theta\right) \ , \tag{18}$$

with solutions

$$\theta = 0\ , \quad \theta = \pi\ , \quad \cos\theta = \frac{1}{1+\xi} \equiv c_0\ , \qquad \xi \equiv \frac{\alpha_1' - \alpha_c}{2\alpha_2}\ . \tag{19}$$

Since $\alpha_2 > 0$ one can see that $V(\pi) > V(0)$ always, and so $\theta = \pi$ never corresponds to a minimum of the potential. For $\alpha_1' < \alpha_c$, the $\cos\theta = c_0$ solution is never physical, as in that case $\xi < 0$ and $c_0 > 1$. However for $\alpha_1' > \alpha_c$ then $\xi > 0$ and this must be the true vacuum since we have seen that the $\theta = 0$ extremum of the potential becomes unstable (a maximum). So the solution is:

$$\theta = \begin{cases} 0 & \alpha_1' \le \alpha_c \\ \theta_0 \equiv \cos^{-1}\left[\frac{\alpha_2}{\alpha_2 + (\alpha_1' - \alpha_c)}\right] & \alpha_1' \ge \alpha_c \end{cases} \tag{20}$$

(e) If we substitute our nontrivial vacuum eq. (16) into the kinetic term with the $SU(2)\times U(1)$ gauge boson fields we get

$$\mathcal{L} = \frac{F_\pi^2}{4}\mathrm{Tr}\left[\left((ig_2 W_a[T_a,\Sigma] + ig_1 B[Y,\Sigma]\right)\left((ig_2 W_a[T_a,\Sigma] + ig_1 B[Y,\Sigma]\right)^\dagger\right] \equiv \frac{1}{2}A_\alpha M^2_{\alpha\beta}A_\beta\ , \tag{21}$$

with

$$A_\alpha = \begin{cases} W_a & \alpha = a = 1,2,3 \\ B & \alpha = 4 \end{cases} \tag{22}$$

with

$$M^2 = \frac{F_\pi^2}{4}\begin{pmatrix} g_2^2 2(1-c_0) & & & \\ & g_2^2 2(1-c_0) & & \\ & & g_2^2 s_0^2 & -g_1 g_2 s_0^2 \\ & & -g_1 g_2 s_0^2 & g_1^2 s_0^2 \end{pmatrix} \tag{23}$$

where $c_0 = \cos\theta_0$ and $s_0 = \sin\theta_0$ where θ_0 is our nontrivial solution for the vacuum alignment found above in eq. (20).

The above expression *does not* in general look like the SM mass matrix for the W and Z which we saw in class unless we have

$$2(1-c_0) = s_0^2 = \frac{v^2}{F_\pi^2}\ , \qquad v = 246\ \text{GeV}\ . \tag{24}$$

...which does hold however if $\theta_0 = (v/F_\pi) \ll 1$. In other words, the gauge bosons look standard if the compositeness scale F_π is very high compared to the weak scale, which would require tuning α_1' to be very close to the critical coupling $\alpha_c = (3\alpha_2 + \alpha_1)$.

(f) The potential we have computed has two free parameters, F_π and α_1'. If we take $\alpha_1' \to \alpha_c$ (where α_c in eq. (15) is known in terms of the known SM couplings $g_{1,2}$), and we insist that the W, Z masses have their conventional values, then we have fixed both of these parameters. Thus the potential is completely determined and we should be able to compute the Higgs boson mass.

The Higgs boson mass is found by writing $\theta = (v+h)/F_\pi$, expanding the potential to second order in h about its minimum:

$$M_h^2 = \left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta \to \theta_0} \tag{25}$$

With $(\alpha_1' - \alpha_c) \ll 1$ the angle at the minimum is given by

$$\theta_0 = \cos^{-1}\left[\frac{\alpha_2}{\alpha_2 + (\alpha_1' - \alpha_c)}\right] \simeq \sqrt{\frac{2(\alpha_1 - \alpha_c)}{\alpha_2}} \ . \tag{26}$$

Since we want to keep the W, Z masses fixed at their physical values we set

$$F_\pi = \frac{v}{\theta_0} \ , \qquad v = 246 \text{ GeV} \ . \tag{27}$$

Thus we get

$$M_h^2 \quad = \quad \lim_{\alpha_1' \to \alpha_c} \left.\frac{1}{F_\pi^2}\frac{\partial^2 V}{\partial \theta^2}\right|_{F_\pi \to v\theta_0, \theta \to \theta_0} = c\frac{\alpha_2}{4\pi}v^2 = c\frac{\alpha}{4\pi \sin^2\theta_w}v^2 \ . \tag{28}$$

and so our answer for the Higgs mass is

$$M_h = v\sqrt{c\frac{\alpha}{4\pi \sin^2\theta_w}} = 145 \text{ GeV} \ , \tag{29}$$

where I used $v = 246$ GeV, $\sin^2\theta_w = 0.23$, $\alpha = 1/137.$, and the result for $c = 137.$ from eq. (4). A bit too heavy! The real Higgs mass is 125 GeV, but it is cool that we can compute it. All we need is a factor of $\frac{3}{4}$ in front of our expression for M_h^2...