1. Wilson Loop in External Field

What we want to calculate in this problem is the geometric law obeyed by the expectation value of the planar Wilson loop, $\langle W_C \rangle$, with a transverse external random gaussian magnetic field $\langle B_z(\vec{r}) B_x(\vec{r}') \rangle = \sigma^2 \delta^2(\vec{r} - \vec{r}')$. Here $\vec{r}$ denotes a point in the plane of the contour for W_C. Addressing the note, we consider the unnormalized probability distribution for B_z, $P_B = exp\big[-k \int d^2 r B_z(r)^2 \big]$, such that

$$\langle B_z(x) B_z(y) \rangle = \frac{\int [dB] e^{-k \int d^2 r B_z(r)^2} B_z(x) B_z(y)}{\int [dB] e^{-k \int d^2 r B_z(r)^2}}, \tag{1}$$

$$= \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \frac{\int [dB] e^{-\int d^2 r (k B_z(r)^2 - J(r) B_z(r))}}{\int [dB] e^{-k \int d^2 r B_z(r)^2}} |_{J=0}. \tag{2}$$

Completing the square and taking the functional differentiation gives,

$$\langle B_z(x) B_z(y) \rangle = \frac{1}{2k} \delta^2(x-y) \quad \Rightarrow \quad k = \frac{1}{2\sigma^2}. \tag{3}$$

Now, back to the main problem. Let us denote $\int_\Sigma$ as the surface integral over the region Σ bounded by the loop contour, C, and $\oint_C$ as the line integral around C. Calculating the expectation value of the Wilson loop is then a matter of calculating

$$\langle W_C \rangle = \frac{\int [dB] e^{-k \int d^2 r B_z(r)^2} e^{ig \oint_C A_\mu dl^\mu}}{\int [dB] e^{-k \int d^2 r B_z(r)^2}} = \frac{\int [dB] e^{-k \int d^2 r B_z(r)^2} e^{ig \int_\Sigma d^2 r B_z(r)}}{\int [dB] e^{-k \int d^2 r B_z(r)^2}}. \tag{4}$$

In the last expression, the line integral of A_μ around the contour C was exchanged for the surface integral of B_z. This can easily be checked using the standard expressions relating A_μ and B. It is now useful to split the integral in the exponential arising from the probability distribution into two area integrals $\int d^2 r B_z(r)^2 \to \int_\Sigma d^2 r B_z(r)^2 + \int_{x \notin \Sigma} d^2 r B_z(r)^2$, which after completing the square in the exponential gives

$$\langle W_C \rangle = \frac{\int [dB] e^{-\int_\Sigma d^2 r (k B_z(r)^2 - ig B_z(r))}}{\int [dB] e^{-k \int_\Sigma d^2 r B_z(r)^2}} = \frac{\int [dB] e^{-k \int_\Sigma d^2 r [(B_z(r) - i \frac{g}{2k})^2 + \frac{g^2}{4k^2}]}}{\int [dB] e^{-k \int_\Sigma d^2 r B_z(r)^2}}. \tag{5}$$

Performing the integration then yields, with $\int_\Sigma d^2 r = \mathcal{A}_\Sigma$

$$\langle W_C \rangle = e^{-\frac{g^2}{4k} \mathcal{A}_\Sigma}. \tag{6}$$

We, thus, find an area law, which indicates that we are in a confined phase!

2. Two-Flavor NJL

In this problem, we want to consider a generalization of the NJL-model considered in the handout. Instead of a single flavor model with a $U(1)_A$ symmetry, we want to consider a chiral two-flavor model with a $U(2)_L \times U(2)_R$ symmetry. The fermion condensate will spontaneously break $SU(2)_L \times U(2)_R \to SU(2)_V$. We will denote throughout the problem

$\Phi = \sigma \mathbb{1}_2 + i\pi_a\tau_a$, where τ_a are Pauli matrices and π_a, σ are real scalar fields. We will start with the Lagrangian density

$$\mathcal{L} = \frac{N}{4g}\text{Tr}\,\Phi^\dagger\Phi + \sum_{n=1}^{N}\sum_{i=1,2}(\bar{\psi}_{L,ni}\partial\!\!\!/\psi_{L,ni} + \bar{\psi}_{R,ni}\partial\!\!\!/\psi_{R,ni}) + \sum_{n=1}^{N}\sum_{i,j=1,2}(\bar{\psi}_{L,ni}\Phi_{ij}\psi_{R,nj} + \text{h.c.}). \quad (7)$$

a. Symmetry and Transformations

We can read off the symmetries of the theory from noting that there is, in addition to the chiral flavor symmetry, a $U(N)$ symmetry rotating the fermions as in the single flavor case $\psi_n \to U_{nm}\psi_n$. So the total symmetry of the Lagrangian is just $U(N)\times SU(2)_L\times SU(2)_R$. Φ is uncharged under the global $U(N)$. The action of the $SU(2)$'s on the fermions is given by

$$SU(2)_L: \quad \psi_L \to e^{i\theta_a\tau_a}\psi_L, \quad \psi_R \to \psi_R, \quad (8)$$
$$SU(2)_R: \quad \psi_R \to e^{i\tilde{\theta}_a\tau_a}\psi_R, \quad \psi_L \to \psi_L. \quad (9)$$

Looking at the interaction term, we can then deduce that the scalars then must transform as

$$SU(2)_L: \quad \Phi \to e^{i\theta_a\tau_a}\Phi, \quad (10)$$
$$SU(2)_R: \quad \Phi \to \Phi e^{-i\tilde{\theta}_a\tau_a}. \quad (11)$$

Unpacking $\Phi = \sigma\mathbb{1}_2 + i\pi_a\tau_a$ and following the infinitesimal versions of the above transformations

$$SU(2)_L: \delta\Phi = i\sigma\theta_a\tau_a - \theta_a\pi_b\tau_a\tau_b \quad \Rightarrow \quad \delta\sigma = i\theta_a\tau_a, \quad \delta\pi_b = i\theta_a\pi_b\tau_a, \quad (12)$$
$$SU(2)_R: \delta\Phi = -i\sigma\tilde{\theta}_a\tau_a + \tilde{\theta}_a\pi_b\tau_b\tau_a \Rightarrow \delta\sigma = -i\tilde{\theta}_a\tau_a\sigma, \, \delta\pi_b = -i\tilde{\theta}_a\pi_b\tau_a. \quad (13)$$

Note in the last line, care must be taken to order the τ's correctly. That is, since $\tilde{\theta}_a$ and π_b are just scalars $\delta(\pi_b\tau_b) = -i(\pi_b\tau_b)\tilde{\theta}_a\tau_a = -i\tau_b(\tilde{\theta}_a\pi_b\tau_a)$. Combining the action on all of infinitesimal transformationsL

$$SU(2)_L: \quad \delta\psi_L = i\theta_a\tau_a\psi_L, \quad \delta\psi_R = 0, \quad \delta\sigma = i\sigma\theta_a\tau_a, \quad \delta\pi_b = i\theta_a\tau_a\pi_b, \quad (14)$$
$$SU(2)_R: \quad \delta\psi_L = 0, \quad \delta\psi_R = i\tilde{\theta}_a\tau_a\psi_R, \quad \delta\sigma = -i\sigma\tilde{\theta}_a\tau_a, \quad \delta\pi_b = -i\tilde{\theta}_a\pi_b\tau_a. \quad (15)$$

b. Integrating Out Φ

We start with eq. 7 and derive the equations of motion for Φ. To be explicit, we can expand $\Phi = \sigma\mathbb{1}_2 + i\pi_a\tau_a$, such the $\mathcal{L}$ (suppressing the summations)

$$\mathcal{L} = \frac{N}{2g}(\sigma^2 + \pi_a^2) + \bar{\psi}_L\partial\!\!\!/\psi_L + \bar{\psi}_R\partial\!\!\!/\psi_R + (\bar{\psi}_L(\sigma\mathbb{1}_2 + i\pi_a\tau_a)\psi_R + \bar{\psi}_R(\sigma\mathbb{1}_2 - i\pi_a\tau_a)\psi_L)$$

Varying $\mathcal{L}$ with respect to σ and π_b

$$\frac{\delta\mathcal{L}}{\delta\sigma} = 0, \quad \Rightarrow \quad \sigma = -\frac{g}{N}\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right) = -\frac{g}{N}\bar{\psi}\psi, \quad (16)$$
$$\frac{\delta\mathcal{L}}{\delta\pi_b} = 0, \quad \Rightarrow \quad \pi_b = \frac{ig}{N}\left(\bar{\psi}_R\tau_b\psi_L - \bar{\psi}_L\tau_b\psi_R\right) = -\frac{ig}{N}(\bar{\psi}\tau_b\gamma_5\psi). \quad (17)$$

Now integrating out Φ, we perform the Gaussian path integral over σ and π_a. As there is no kinetic term for the auxiliary scalars, this is just replacing σ and π_a by their equations of motion. Substituting back in to the Lagrangian we find

$$\mathcal{L}_f = \bar{\psi}\partial\!\!\!/\psi - \frac{g}{2N}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau_a\psi)^2\right]. \tag{18}$$

c. Vacuum Structure

Exploring the non-trivial vacuum structure. From the equations of motion, we note that rewriting the path integral

$$\mathcal{Z} = \mathcal{N}\int D\psi D\bar{\psi}e^{-S} = \mathcal{N}'\int D\psi D\bar{\psi}\int D\sigma D\vec{\pi}e^{-S'}, \tag{19}$$

the scalar one point functions are analogous to the $U(1)$ NJL model:

$$\langle\sigma\rangle = -\frac{g}{N}\sum_{n,i}\langle\bar{\psi}_{n,i}\psi_{n,i}\rangle, \quad \langle\pi_a\rangle = -\frac{g}{N}\sum_{n,ij}\langle\bar{\psi}_{n,i}i\gamma_5(\tau_a)_{ij}\psi_{n,j}\rangle. \tag{20}$$

Our next step in probing the vacuum structure is integrating out the fermions and generate a fermionic functional determinant. First, we need to rewrite the Lagrangian density in terms of fermion bilinears

$$\mathcal{L} \to \mathcal{L}' = \frac{N}{2g}(\sigma^2 + \pi_a^2) + \sum_{n,ij}\bar{\psi}_{n,i}(\delta_{ij}\partial\!\!\!/ + \delta_{ij}\sigma + i\pi_a(\tau_a)_{ij}\gamma_5)\psi_{n,j}. \tag{21}$$

Integrating out the fermions gives the bosonized Lagrangian density as

$$\mathcal{L}_\Phi = N\Big(\frac{1}{2g}(\sigma^2 + \pi_a^2) - \sum_{ij}\text{Tr }\ln(\delta_{ij}(\partial\!\!\!/ + \sigma) + i\pi_a(\tau_a)_{ij}\gamma_5)\Big). \tag{22}$$

The path integral is now only over σ and π_a, and with the overall N prefactor the $N \to \infty$ limit localizes the theory on the saddles of the action. In this limit, we want to find the non-trivial vacuum structure corresponding to $\langle\sigma\rangle = f \neq 0$ and $\langle\pi_a\rangle = 0$. Finding the saddles

$$\frac{\partial S_\Phi}{\partial\sigma}\Big|_{\langle\sigma\rangle,\,\langle\pi_a\rangle} = N\Big(\frac{\langle\sigma\rangle}{g} - \sum_{ij}\text{Tr}\,\frac{\delta_{ij}}{\delta_{ij}(\partial\!\!\!/ + \langle\sigma\rangle) + i\langle\pi_a(\tau_a)_{ij}\rangle\gamma_5}\Big), \tag{23}$$

$$= N\Big(\frac{f}{g} - 2\text{Tr}\,\frac{1}{\partial\!\!\!/ + f}\Big) = 0. \tag{24}$$

So, we see that the saddle points are determined by an equation that looks like the $U(1)$ case apart from the factor of 2 coming from the sum over flavor indices. We have in essence guaranteed that by picking the particular vacuum structure that we want to probe. We thus end up with, structurally, the same conditions

$$f = 0, \quad \text{or} \quad \frac{1}{2g} = 4\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2 + f^2} \equiv I \tag{25}$$

Since, we wanted to see the vacuum with $f \neq 0$, let us look at the integral equation using dimensional regularization. We can use our standard techniques, letting $D = 3 + \epsilon$ and taking the limit $\epsilon \to 0$ after integration, we find that noting the beta function is defined in terms of Euler Gamma functions as $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, I evaluates to

$$I = \frac{4\mu^{3-D} f^{D-2}}{2^D \pi^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} B\left(\frac{D}{2}, 1 - \frac{D}{2}\right) \xrightarrow{\epsilon \to 0} -\frac{f}{\pi}\Big(1 + \frac{\epsilon}{2}\big(2 - \log\big(\frac{f^2 e^{\gamma}}{\mu^2 \pi}\big)\big) + \mathcal{O}(\epsilon^2)\Big). \tag{26}$$

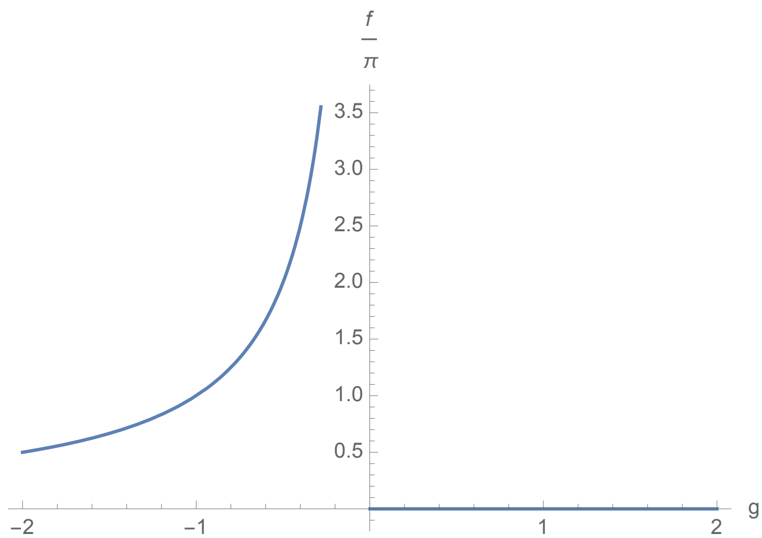
So, renormalizing with $\overline{\text{MS}}$, we find

$$\frac{f}{\pi} = -\frac{1}{2g(1 - \epsilon + \epsilon \log\left(\frac{f}{\mu}\right))} \xrightarrow{\epsilon \to 0} -\frac{1}{2g} \tag{27}$$

Is this a true minimum? We need to check

$$\frac{\partial^2 S_\Phi}{\partial \sigma^2}\Big|_{\langle \pi_a \rangle = 0,\, \langle \sigma \rangle = 0} = \frac{N}{g} > 0. \tag{28}$$

Using the above form for $\frac{f}{\pi} > 0$, we again see that for $g < 0$ the $\langle \sigma \rangle = 0$ vacuum is unstable and the $\langle \sigma \rangle = f$ vacuum is preferred.



d. GS Symmetry and NGBs

To answer the question of how many NG modes we should have, we need to find the dimension of the coset space for the preserved symmetry. For a generic spontaneous symmetry breaking of a symmetry group G to $H \subset G$, the number of NG modes corresponds to the number of generators broken in the process, which corresponds to dim G/H. In this case, since the $\langle \sigma \rangle = f$, $\langle \pi_a \rangle = 0$ vacuum preserves $SU(2)_V \subset SU(2)_L \times SU(2)_R$, we can perform this computation straightforwardly. We know that $SU(2)$ has 3 generators, and so the quotient group $(SU(2)_L \times SU(2)_R)/SU(2)_V$ has $(3+3) - 3 = 3$ generators. The Nambu-Goldstone modes associated with the broken generators are exactly the π_a's!