1. Vortex-Antivortex Pair:
Starting from the problem statement, we have a complex scalar field in 2+1 dimensions invariant under a global U(1) with a vev $\langle\phi\rangle = f/\sqrt{2} \equiv v$. Thus our Lagrangian density looks like
$$\mathcal{L} = \frac{1}{2}|\partial\phi|^2 - \lambda\Big(|\phi|^2 - v^2\Big)^2. \tag{1}$$
For a solitonic configuration, i.e. a single vortex, we know that the global U(1) means that the maps at spatial infinity $\phi_\infty : S^1 \to S^1$ are quantized by the Pontrjagin index $n \in \mathbb{Z}$. An appropriate ansatz for the asymptotic behavior is then $\phi_\pm \sim v$. We then have, adopting polar coordinates (r, θ) on the spatial slices, $\partial\phi_\pm \sim \pm v/r$. Here we distinguish the vortex from anti-vortex by $+$ and $-$ labels respectively. From the form of $\mathcal{L}$, we can find the Hamiltonian $\mathcal{H}$ by performing the usual Legendre transformation
$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\partial_t\phi - \mathcal{L} \tag{2}$$
We can then determine the energy by integrating $\mathcal{H}$ over the spatial directions. In order to do so, we put the system in a box of size R. We then find that
$$E_+ = \int^R d^2x\Big(\frac{1}{2}|\partial_t\phi|^2 + \frac{1}{2}|\nabla\phi|^2 + \lambda\big(|\phi|^2 - v^2\big)^2\Big) \sim v^2 \log R. \tag{3}$$
We see that the energy is logarithmically divergent with the size of the box. If we consider a vortex-antivortex pair, we will find the energy to be finite. However a static vortex-antivortex configuration will only be an approximate solution to the equations of motion, and then when they are only far apart.

⌜ Somewhat easier to picture is the case for the instanton where the field asymptotically tunnels the neighboring vacuum with no energy cost. In that case, adding in a ‘nearby’ anti-instanton renders the solution void, but for widely separated instanton-anti-instanton pairs (often called the Dilute Instanton Gas), the deviation from being solution is small and localized near where the two configurations are stitched together. ⌟

Lets consider the pair labeled $\phi_+(r)$, $\phi_-(r)$ separated by some distance $R = r_+ - r_-$, which for simplicity and validity of our analysis to be much larger than the typical size of the vortex labeled by a. Then, defining the pair by $\varphi(r) = \phi_+(r + r_+)\phi_-(r + r_-)$, we see that given the asympotic behavior of ϕ and $\partial\phi$ as described above, we now have that
$$\varphi \sim v, \quad \partial\varphi \sim 0. \tag{4}$$
So, the interaction energy associated to the pair is given by
$$E_\pm \sim v^2 \int d^2x \frac{1}{r^2} \sim v^2 \log\frac{R}{a}. \tag{5}$$
So, we can see that there is a potential $V(R) \sim \log R$, which implies an attractive 2+1 dimensional Coulombic $(1/R)$ type force, between the pair. The breakdown of this configuration occurs when we see that the assumption that the separation between vortex cores being much larger than the typical size of the profile is no longer valid. This occurs when the distance between the centers of the vortex and antivortex get within a few of the scalar Compton wavelengths apart.

2. Wilson Lines and Loops:

   a. Gauge transformation of Wilson line:
   In the following, we will assume that $G = SU(N)$. What we would like to show here is that under the gauge transformation $A \to UAU^\dagger + \frac{i}{g}UdU^\dagger$

$$W_C(y,x) \to U(y)W_C(y,x)U^\dagger(x), \tag{6}$$

   where $W_C(y,x) \equiv P\Big[exp\{ig\int_x^y A_\mu dx^\mu\}\Big]$. Consider how the integrand in the exponential changes under gauge transformation and use that $A = A^aT^a$ is a Lie algebra valued one-form gauge field. Denote the gauge transformed Wilson loop $\tilde{W}_C$, which is given by

$$\tilde{W}_C = P\Big[e^{ig\int UA_\mu U dx^\mu - \int U\partial_\mu U^\dagger dx^\mu}\Big]. \tag{7}$$

   For the purposes of illustration, consider the infinitesimal contour $W_C(x+\epsilon,\, x)$, and Taylor expand. For a single displacement,

$$\tilde{W}_C = 1 + ig\epsilon^\mu U(x)A_\mu U^\dagger(x) - \epsilon^\mu U(x)\partial_\mu U^\dagger(x) + \mathcal{O}(\epsilon^2), \tag{8}$$
$$= (U(x+\epsilon)U^\dagger(x) + \mathcal{O}(\epsilon^2)) + (ig\epsilon^\mu U(x+\epsilon)A_\mu U^\dagger(x) + \mathcal{O}(\epsilon^2)) + \mathcal{O}(\epsilon^2), \tag{9}$$

   where in the last line, we used unitarity of the U's, and replaced both $(1+\epsilon^\mu\partial_\mu)U(x)$ and $\epsilon^\mu U(x)$ by $U(x+\epsilon)$ while suppressing further factors of $\mathcal{O}(\epsilon^2)$. From here it is a simple rewriting to see that the infinitesimal form is

$$\tilde{W}_C = U(x+\epsilon)(1 + ig\epsilon^\mu A_\mu)U^\dagger(x) \quad \Rightarrow \quad \tilde{W}_C = U(x+\epsilon)W_C U^\dagger(x). \tag{10}$$

   Building a finite path is simply chaining together the path ordered set of infinitesimal lines. The gauge transformation of the finite path is then easily obtained by breaking up (y,x) into n infinitesimal lines such that

$$\begin{aligned}\tilde{W}_C(y,x) &= \tilde{W}(y, y-\epsilon_n)\tilde{W}_C(y-\epsilon_n, y-\epsilon_{n-1})\cdots\tilde{W}_C(x+\epsilon_1, x) &(11)\\ &= U(y)W_C(y, y-\epsilon_n)\cdots W_C(x+\epsilon_1, x)U^\dagger(x)\\ &= U(y)W_C(y,x)U^\dagger(x). &(12)\end{aligned}$$

   b. Square loop:
   Let us consider the Wilson Loop, where we construct the square path in spacetime around the square loop $z^\mu \to z^\mu + an^\mu \to z + a(n^\mu + n^\nu) \to z + an^\nu \to z$. In mathematical literature, this object is called the *gauge holonomy* and is an analog to measuring the Riemann curvature of the gauge bundle (i.e. how parallel transport around a closed loop in the base space fails to return the gauge connection back to its original value in the fibre). For simplicity call the plane of the loop the $1-2$ plane, denote $an^\nu = \hat{a}^1$ and $an^\mu = \hat{a}^2$. The Wilson loop is then

$$W_C(z,z) = W_C(z,\, z+\hat{a}^2)W_C(z+\hat{a}^2,\, z+\hat{a}^1+\hat{a}^2)W_C(z+\hat{a}^1+\hat{a}^2, z+\hat{a}^1)W_C(z+\hat{a}^1,\, z). \tag{13}$$

   Of course will be interested in the trace of this object, but for now we can easily compute by applying the logic of the previous section. Using the anti-hermicity of each link $W_C^\dagger(y,x) = W_C(x,y)$, we find that expanding A_μ in the infinitesimal displacement $\hat{a}$ allows us to write

$$W_C(z+\hat{a}, z) = e^{ig\hat{a}^i A_i(z+\frac{\hat{a}}{2}) + \mathcal{O}(a^3)}. \tag{14}$$

Since, we are interested in the leading correction, we will drop terms of $\mathcal{O}(a^3)$ in the exponentials. Now, keeping in mind the orientation of the path and which axis the displacement is on, we see that

$$W_C(z,z) = e^{-igaA_2(z+\frac{\hat{a}^2}{2})}e^{-igaA_1(z+\hat{a}^2+\frac{\hat{a}^1}{2})}e^{igaA_2(z+\hat{a}^1+\frac{\hat{a}^2}{2})}e^{igaA_1(z+\frac{\hat{a}^1}{2})}, \tag{15}$$

$$= e^{iga^2F_{12}}, \tag{16}$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1 - ig[A_1,\, A_2]$ and the ∂_i's emerge from seeing that the $\mathcal{O}(g)$ terms in expanding the product of the exponentials is just the anti-symmetric combination of difference operators around the loop. Now, in taking the trace of $W_C(z,z)$ we need to be sure to include the contribution from traversing the loop in the opposite direction, $W_{-C}(z,z)$. That is, denoting $W_P = W_C(z,z) + W_{-C}(z,z)$, and expanding for small a,

$$\mathrm{Tr}\,(W_P) = Tr(e^{iga^2F_{12}} + e^{-iga^2F_{12}}) \tag{17}$$

$$= 2N - g^2a^4\mathrm{Tr}\,F_{12}^2 + \dots. \tag{18}$$

This confirms the $\mathcal{O}(a^4)$ behavior indicated in the problem.

3. Derrick's Theorem

 a. Energy conditions:

We would like to consider the massless interacting scalar, ϕ, in some arbitrary potential $V(\phi)$ in $(D+1)$-dimensions such that

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi). \tag{19}$$

The energy of this system evaluated on some solitonic solution, which I will also call ϕ, is

$$E_\phi = \int d^Dx\Big(\frac{1}{2}|\partial_t\phi|^2 + \frac{1}{2}|\nabla\phi|^2 + V(\phi)\Big). \tag{20}$$

Take eq. (20) and compute using the rescaled configuration and let $y \equiv sx$

$$E_\phi(s) = \frac{1}{2}\int d^Dx(|\partial_t\phi(sx)|^2 + |\nabla\phi(sx)|^2) + \int d^DxV(\phi(sx)), \tag{21}$$

$$= \frac{s^{2-D}}{2}\int d^Dy(\partial\phi(y))^2 + s^{-D}\int d^DyV((\phi(y)) \equiv s^{2-D}T + s^{-D}U, \tag{22}$$

where we defined the kinetic (T) and potential energy (U) of the configuration in the last line. Now, what conditions are needed in order for ϕ to be a solution? First, a solitonic solution is necessarily a field configuration that extremizes the energy,

$$\partial_s E_\phi(s) = 0 \quad \Rightarrow \quad T = -\frac{D}{(D-2)s^2}U \tag{23}$$

In order for ϕ to be a *stable* solution, we need to check the second derivative. Evaluating at the extremum in eq. (23), we find that this requires

$$\partial_s^2 E_\phi(s) = -2(D-2)s^{-D}T < 0. \tag{24}$$

Given that $T > 0$ and the dimensionality of the spacetime $d = (D+1) > 0$, eq. (23) and eq. (24) show that the only stable solutions are in $0 < d < 1+1$ (or $0 \leq D < 1$).

b. Multiple scalars:

If we included multiple scalars ϕ_i arranged in a solitonic configuration as above, we should not expect the arguments to change. Consider eq. (20) with $\phi \to \phi_i$

$$E_{\vec{\phi}} = \int d^D x \sum_i \Big(\frac{1}{2}|\partial_t \phi_i|^2 + \frac{1}{2}|\nabla \phi_i|^2 + V(\phi_i)\Big). \tag{25}$$

If we affect the same rescaling, then again

$$E_{\vec{\phi}}(s) = s^{2-D} T + s^{-D} U. \tag{26}$$

Here T and U denote the total kinetic and total potential energy of the system. That is, the scaling behavior of any individual contribution to the total kinetic or potential energy has to scale the same way as any other contribution. We can employ the same logic as part (a) to arrive at the same conclusion.

c. Implication for vortices and monopoles:

The above story has drastic implication for the existence of vortex, i.e. isolated solitonic, configurations. There are no localized vortices in $D > 1$ as a consequence of eq. (23) and eq. (24). This is Derrick's Theorem.

Let us now consider the case of a monopole. Turning on a gauge field A, we need now need to promote the ordinary derivatives to covariant derivatives $\partial_\mu \phi \to D_\mu \phi = \partial_\mu \phi + igA_\mu \phi$. Considering the scalar part of the theory

$$\tilde{\mathcal{L}}_\phi = \frac{1}{2}|D\phi|^2 - V(\phi) = \frac{1}{2}|\partial\phi|^2 + |A\phi|^2 + 2A^\mu \phi \partial_\mu \phi - V(\phi). \tag{27}$$

The cross term $A^\mu \phi \partial_\mu \phi$ gives an additional contribution to the energy that will allow for a finite sized, stable, time independent solution for $d > 1 + 1$. Considering full Lagrangian density by including the Yang-Mills term, $\tilde{\mathcal{L}}_\phi + \mathcal{L}_{YM}$, we would find that the energy has scaling behavior

$$E(s) = s^{4-D} E_A + s^{2-D} T_A + \mu^{-D} U, \tag{28}$$

where T_A denotes the kinetic energy in addition to any contributions from A with scaling s^{2-D} and E_A denotes the novel scaling contributions due to A. Following an analysis similar to part (a) would show that the new scaling behavior gives us the freedom to find stable, time independent, spatially localized solutions in $D \geq 1$.