1. Adjoint Rep:

   a) We start with a Lie algebra $\mathfrak{g}$ with generators labeled by $T_a$. By definition we have the Lie bracket

$$[T_a,\, T_b] = i f_{abc} T_c, \tag{1}$$

   where $f_{abc}$ are the structure constants. In mathematical literature, this structure is referred to as the adjoint endomorphism $ad_x : \mathfrak{g} \to \mathfrak{g}$ acting as $ad_x(y) = [x,\, y]$ for $y \in \mathfrak{g}$. A useful relation to use here is the Jacobi identity

$$[T_a,\, [T_b,\, T_c]] + [T_c,\, [T_a,\, T_b]] + [T_b,\, [T_c,\, T_a]] = 0. \tag{2}$$

   Feeding in the algebra satisfied by the generators we see that eq. (2) implies

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0. \tag{3}$$

   Defining the components of a matrix $(t_a)_{bc} \equiv -i f_{abc}$, we see that eq. (2) can be rewritten to be

$$[t_a,\, t_b] = i f_{abc} t_c. \tag{4}$$

   Thus, the structure constants furnish for us a representation. Note that the matrices $t_a$ are $d \times d$ dimensional, where $d$ is the number of generators. This can be easily seen from eq. (1).

   b) Defining $\Phi = \phi_a T_a$, we can affect the transformation $\delta_a \Phi$ as specified in the problem statement by

$$\delta_a \Phi = i[T_a, \Phi] \quad \Rightarrow \quad \delta_a \Phi = -f_{abc} T_c \phi_b. \tag{5}$$

   Performing the transformation on $\phi_a$ instead and using the definition of $(t_a)_{bc} = -i f_{abc}$

$$\delta_a \phi_b = i(t_a)_{bc} \phi_c = f_{abc} \phi_c. \tag{6}$$

   How are these two equivalent? Lets think about eq. (6) in terms of a transformation on $\Phi$. That is,

$$(\delta_a \phi_b) T_b = f_{abc} \phi_c T_b \quad \Rightarrow \quad \delta_a \phi_b T_b = -f_{acb} \phi_c T_b = \delta_a \Phi, \tag{7}$$

   where the last expression required use of the fact that $f_{abc}$ is antisymmetric.

2. Minimal GUTs: SU(5)

   a) Starting at the SU(5) breaking scale, $M_{GUT}$, we need to run the various couplings down to the relevant scale, $M_Z$, so that the weak mixing angle,

$$\sin^2 \theta_W = \frac{g'^2}{g_2^2 + g'^2} \equiv x_W^2, \tag{8}$$

   can be compared to experimental values. Standard values relevant to this problem can be found in, e.g., Quigg §7.7 or freely available in the PDG. For our purposes lets take the value to which we compare as $x_W^2(M_Z) \approx 0.231$.

To find the GUT prediction, we begin with the embedding of the Standard Model, SU(3)×SU(2)×U(1)$_Y$, in the **5** of SU(5). SU(3) inhabits the upper 3×3 block; SU(2) sits in the lower 2×2 block; U(1)$_Y$ is the diagonal element

$$\frac{Y}{2} = \text{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right). \tag{9}$$

By the statement of the problem, at the GUT scale all of the couplings are unified, and so $\alpha_5 = \alpha_3 = \alpha_2 = \alpha_1$, where $\alpha_i = \frac{g_i^2}{4\pi}$.

⌜ **N.B.** The hypercharge coupling, $g'$ is, due to the choice of normalization of the generator $Y$, different from $g_1$ by a numerical factor

$$\frac{1}{2}g'Y = g_1 T_{24} \quad \Rightarrow \quad g' = \sqrt{\frac{3}{5}}g_1.$$

We can see this easily by recalling that the electric charge generating the U(1)⊂SU(2)× U(1) can be written as $Q = T_3 + \xi T_{24}$, where $T_3$ is an SU(2)$_L$ generator. A standard calculation gives,

$$\sum Q^2 = (1+\xi^2)\sum T_3^2 \quad \Rightarrow \quad \xi^2 = \frac{3}{5},$$

where we used that $\sum T_3^2 = \frac{1}{2}$ and $\sum Q^2 = \frac{4}{3}$. ⌟

Now that we have settled that, we can go about finding $x_W^2(M_Z)$. First note that before any breaking of the SU(5) occurs, the unified couplings give $\sin^2\theta_W(M_{GUT}) = \frac{3}{8}$. Next, we need the beta functions, $\beta_i$ for the relevant couplings. Since experimentalists have a good precision data from which they extract the Weinberg angle, to get a prediction that would be best compared to experimental values, we should go to higher loop corrections in finding the $\beta_i$'s. However, lets content ourselves with 1-loop for the purposes of illustration. The running of the $\alpha$s is generically given by

$$\frac{1}{\alpha_3(\mu^2)} = \frac{1}{\alpha_3(M^2)} - \frac{1}{4\pi}\left(11 - \frac{4}{3}n_f\right)\log\left(\frac{M^2}{\mu^2}\right) \equiv \frac{1}{\alpha_3(M^2)} + b_3\log\left(\frac{M^2}{\mu^2}\right), \tag{10}$$

$$\frac{1}{\alpha_2(\mu^2)} = \frac{1}{\alpha_2(M^2)} - \frac{1}{4\pi}\left(\frac{22-4n_f}{3}\right)\log\left(\frac{M^2}{\mu^2}\right) \equiv \frac{1}{\alpha_2(M^2)} + b_2\log\left(\frac{M^2}{\mu^2}\right), \tag{11}$$

$$\frac{1}{\alpha_Y(\mu^2)} = \frac{1}{\alpha_Y(M^2)} + \frac{1}{4\pi}\left(\frac{20n_f}{9}\right)\log\left(\frac{M^2}{\mu^2}\right) \equiv \frac{1}{\alpha_Y(M^2)} + b_Y\log\left(\frac{M^2}{\mu^2}\right). \tag{12}$$

where $n_f = 3$ is the number of fermion generations, $\frac{1}{\alpha_Y(m^2)} = \frac{5}{3\alpha_1(m^2)}$. Note, the contribution of the Higgs (as well as any other possible scalar we could posit) has been ignored but could be easily included, if desired. At this point we can easily find the running of $x_W^2$ by multiplying eq. (11) by $\alpha_Y$ as given in eq. (12). That is

$$\frac{\alpha_Y(M^2)}{\alpha_2(M^2)} = x_W^2(M^2) = \frac{3}{8} - \frac{\alpha(M^2)(3b_Y - 5b_2)}{8}\log\left(\frac{M^2}{\mu^2}\right). \tag{13}$$

Plugging in the $b_i$'s from above and running down to the $M = M_Z$, we find

$$x_W^2(M_Z^2) \approx .203. \tag{14}$$

This misses the mark, quoted above, for experimental comparison by about 12%. Note however, that the minimal supersymmetry SU(5) model pushes $x_W^2(M_Z^2) \approx .234$, which sits roughly within the error bars for the experimental result.

b) Flipping the question around; we start with the known values of the couplings $\alpha_1^{-1}(M_Z) \approx 59$, $\alpha_2^{-1}(M_Z) \approx 30$, and $\alpha_3^{-1}(M_Z) \approx 9$. We can run eq. (10)-eq. (12) without the explicit demand of unifying at any scale and see if it happens naturally. The plots generated should should exactly that pairs of $\alpha_i$, $\alpha_j$ unify in the range of $\sim 10^{13}$ GeV to $\sim 10^{17}$ GeV, but the three couplings never hit the same value simultaneously at any scale.

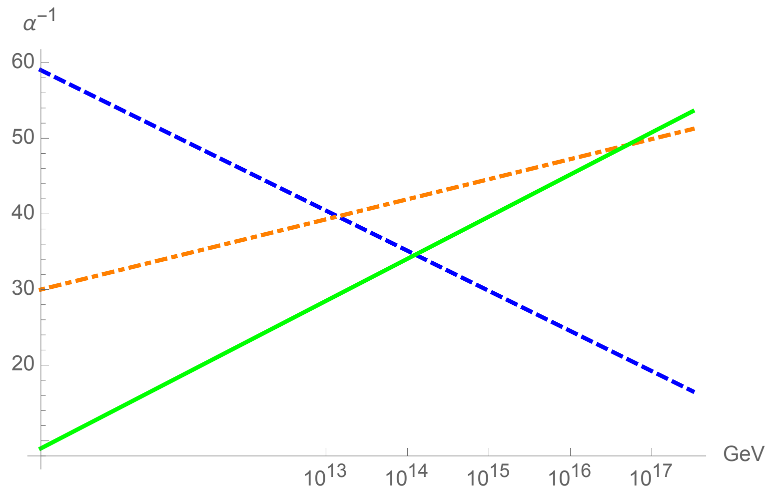


Figure 1: Blue dashed: $\alpha_1$, not $\alpha_Y$. Orange, dot-dashed: $\alpha_2$. Green, solid: $\alpha_3$.

To fix the mismatch in running the couplings from experimentally known values up to the unification scale, we could add in contributions of other particles running in loops to tune the $\beta_i$ appropriately. The Higgs alone doesn't cut it, and we cannot add interacting scalars at will. However, we see that if we take the minimal supersymmetric extension of the SU(5) model, the gauginos and Higgsinos give us the correct number of additional contributions at 1-loop to exactly unify the couplings $M_{GUT} \approx 10^{16}$ GeV with unified coupling $\alpha_{GUT}^{-1} \approx 25$. SUSY also can provide protection against higher order loop corrections, which could potentially make the unification exact. It is assumed here that the SUSY breaking scale is much less than the GUT scale (typically taken on the TeV scale) and that the runnings are effected only by the minimum spectrum all the way up to $M_{GUT}$. These assumptions, while aesthetically pleasing, may be unreasonable to ask of BSM physics.