Physics 572

Problem set #3 **Due Tuesday 5/10/2016**

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1. Compute the expectation value of the Wilson loop $W_C = e^{ig\oint_C A_\mu d\ell^\mu}$ where C is a closed path lying in the $x-y$ plane and A_μ is a $U(1)$ gauge field, assuming that there is a gaussian random field magnetic field pointing in the z direction with $\langle B_z(\mathbf{r})B_z(\mathbf{r}')\rangle = \sigma^2\delta^2(\mathbf{r}-\mathbf{r}')$, the vectors $\mathbf{r}$, $\mathbf{r}'$ lying in the $x-y$ plane; how does $\langle W_C\rangle$ vary as one changes C?

   Note: if a variable x is described by an unnormalized probability distribution $P(x)$, then $\langle f(x)\rangle = \int dx P(x)f(x)/\int dx P(x)$. In this case the unnormalized probability distribution for $B_z(\mathbf{r})$ is given by $\exp(-k\int d^2r B_z(\mathbf{r})^2)$, and the integral over x is replaced by a path integral over $B_z(\mathbf{r})$. You need to figure out how to perform the required integral, and how to relate the constant k to the parameter σ.

2. In class (and in the posted handout) we analyzed a large-N NJL model idimensionally reduced from $4\to 3$ dimensions to illustrate fermion condensation spontaneously breaking a $U(1)_A$ symmetry. A key ingredient was the introduction of an auxiliary field $\phi = (\sigma + i\pi)/\sqrt{2}$. Here I would like you to construct a similar model, but one exhibiting spontaneous breaking of an $SU(2)_L\times SU(2)_R$ symmetry down to $SU(2)_V$ by introducing an auxiliary field $\Phi = \sigma\mathbf{1} + i\pi_a\boldsymbol{\tau}_a$, where $\mathbf{1}$ is the unit matrix, $\boldsymbol{\tau}_a$ are the Pauli matrices with $a = 1,2,3$, and σ, π_a are all **real** scalar fields.

   (a) Start with the Lagrangian

   $$\mathcal{L} = \frac{N}{4g}\mathrm{Tr}\,\Phi^\dagger\Phi + \sum_{n=1}^{N}\sum_{i=1,2}\bar\psi_{ni}\partial\!\!\!/\psi_{ni} + \sum_{n=1}^{N}\sum_{ij=1,2}\left(\bar\psi_{L,ni}\Phi_{ij}\ \psi_{R,nj} + \mathrm{h.c.}\right)\ , \tag{1}$$

   where ψ is a four-component Dirac fermion, $\psi_R = \frac{1+\gamma_5}{2}\psi$, $\psi_L = \frac{1-\gamma_5}{2}\psi$, and the γ-matrices are the usual 4×4 matrices we use in four dimensions. What makes the theory 3-dimensional is that we assume ψ and Φ are independent of one spatial coordinate, and so momentum integrals are over $\int d^3p$ rather than $\int d^4p$. What is the full symmetry of this theory? Show explicitly how the ψ_L, ψ_R, σ and π_a fields transform under infinitesimal symmetry transformations.

   (b) What is the fermion Lagrangian if you integrate out the Φ field?

   (c) Show that in the large N limit there can be a nontrivial vacuum with $\langle\sigma\rangle = f \neq 0$ and $\langle\pi_a\rangle = 0$, using dim reg and $\overline{MS}$ to deal with divergences and renormalization, and determining the condition on g for this ground state to be favored. Sketch f versus g.

   (d) What is the symmetry of this ground state? How many Goldstone bosons should there be? How do they transform under the unbroken symmetry?