Physics 572

Problem set #2 **Due Tuesday 4/26/2016**

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1. Consider a theory of a complex scalar ϕ in 2+1 dimensions with an exact global (not gauged) $U(1)$ symmetry which spontaneously broken by the vacuum expectation value $\langle\phi\rangle = f/\sqrt{2}$ via the potential $V = \lambda(|\phi|^2 - f^2/2)^2$. In this case vortex solutions have infinite energy, but vortex/anti-vortex pairs have finite energy. Estimate the force between a vortex and an anti-vortex widely separated by a distance r. How small does r have to get before your answer is no longer valid?

2. Consider a non-Abelian gauge theory with covariant derivative $D_\mu = \partial_\mu + igA_\mu$ and gauge transformation $A_\mu \to UA_\mu U^\dagger - \frac{i}{g}U\partial_\mu U^\dagger$. If we parametrize a spacetime path C from x to y by the coordinate $z^\mu(\tau)$ with $\tau = [0,1]$, where $z^\mu(0) \equiv x^\mu$ and $z^\mu(1) = y^\mu$, then the Wilson line for that path can be defined as

$$W_C = Pe^{ig\int_c A_\mu dz^\mu} = 1 + ig\int_x^y dz_1^\mu A_\mu(z_1) + (ig)^2\int_x^y dz_1^\mu \int_x^{z_1} dz_2^\nu A_\mu(z_1)A_\nu(z_2)\dot{z}^\mu \dot{z}'^\nu + \dots \tag{1}$$

where integrations are all lie along the same path, and gauge fields are ordered so that as one goes from left to right in the expression $A(z_1)A(z_2)A(z_3)$..., the positions $z_n \equiv z(\tau_n)$ are ordered with $1 \geq \tau_1 \geq \tau_2 \geq \ldots \geq 0$, similar to time ordering.

   (a) Show that under a gauge transformation, the Wilson line transforms as:

$$W_C \to U^\dagger(x)W_C U(y) \tag{2}$$

   (b) Define C to be the infinitesimal square loop in Euclidian spacetime with corners $\{z, z+an^\mu, z+a(n^\mu+n^\nu), z+an^\nu\}$, where a is an infinitesimal length parameter, and n^μ is a unit vector in the μ direction. Compute $\operatorname{Tr} W_C$ to first nontrivial order in a, which you should find to be a^4. How do you interpret your answer?

3. A theory of a scalar field ϕ in D space dimensions has a Lagrangian (mostly minus metric)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{3}$$

where V is an arbitrary function of ϕ, but not its derivatives. Assume that there is a static (time independent) solution to the equations of motion, $\phi(\vec{x})$; then compute the energy of a rescaled configuration $\phi(s\vec{x})$ and see how the energy $E(s)$ varies with scale parameter s.

   (a) What is the condition on $E(s)$ that must necessarily hold if $\phi(\vec{x})$ is really a solution? Is this condition ever satisfied?

   (b) Does your result change if the Lagrangian is a similar function of more than one scalar field ϕ_i?

   (c) In light of your results, comment on the existence of the vortex and monopole solutions we found.