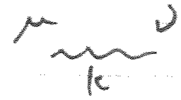
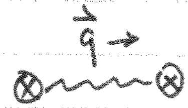
Magnetic moment of the electron (μ)

Take $\mathcal{L} = \mathcal{L}_0^{\text{Maxwell}} + J_\mu A^\mu$

- Consider $J_\mu(x)$ to be static (time-independent), so that $\tilde{J}_\mu(q) \propto \delta(q^0)$
- Work in Feynman gauge  $= \dfrac{-i}{k^2 - i\epsilon} g_{\mu\nu}$

... equivalent to $\partial_\mu \partial^\mu A_\nu = J_\nu$, setting $\partial_\mu A^\mu = 0$.

$i\mathcal{A} =$ 

Current-current correlator $= i^2 J_\mu^*(q) J_\nu(q) \dfrac{-i}{\vec{q}^2 - i\epsilon} g^{\mu\nu}$

Fourier transform: $J_\mu(\vec{x}) J^\mu(\vec{y}) \displaystyle\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \frac{i}{\vec{q}^2 - i\epsilon}$

$$= i \frac{J_\mu(\vec{x}) J^\mu(\vec{y})}{4\pi |\vec{x} - \vec{y}|}$$

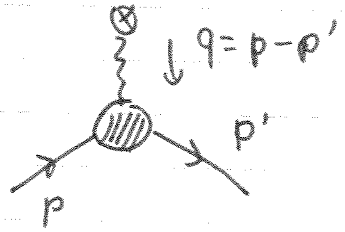
$= \langle f | S - 1 | i \rangle = \langle f | -iH_{int} | i \rangle$

$$H_{int} = -\frac{J_\mu(\vec{x}) J^\mu(\vec{y})}{4\pi |\vec{x} - \vec{y}|}$$

Sign: if $J^0 = \rho$, $\vec{J} = 0$ then $H_{int} = +\dfrac{\rho(\vec{x})\rho(\vec{y})}{4\pi |\vec{x} - \vec{y}|}$

= repulsive Coulomb interaction ✓

Now include electrons and consider



$$= (ie)\left(i\tilde{J}_\mu(q)\right)\left(\frac{-ig^{\mu\nu}}{q^2 - i\epsilon}\right)\bar{u}_{s'}(p')\gamma_\nu u_s(p) + \mathcal{O}(e^3)$$

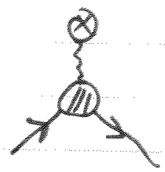
Note that $\tilde{J}_\mu(q) \cdot \frac{1}{q^2 - i\epsilon} = \tilde{A}^{cl}_\mu(q)$, the classical gauge field arising from current J_μ

why? Maxwell eqs: $\partial_\nu \partial^\nu A^{cl}_\mu = -J_\mu$

$$\Rightarrow A^{cl}_\mu = -\frac{1}{\partial_\nu \partial^\nu} J_\mu = \cancel{\#}$$

$$\tilde{A}^{cl}_\mu = \frac{1}{q^2}\tilde{J}_\mu$$

$i\epsilon$ required to make sense of $q^2 = 0$.

Write:  $\equiv i A^{cl}_\mu F^\mu$ (exact)

$$F^\mu = e\,\bar{u}_{s'}(p')\gamma^\mu u_s(p) + \mathcal{O}(e^3) \quad \text{(pert. expansion)}$$

Use Gordon identity:

$$\bar{u}'\gamma^\mu u = -\frac{1}{2m}\bar{u}'(\not{p}'\gamma^\mu + \gamma^\mu \not{p})u$$

← using Dirac eq
$(\not{p} + m)u = 0$
$\bar{u}'(\not{p}' + m) = 0$

and $\gamma^\mu \not{p} = \frac{1}{2}\{\gamma^\mu, \not{p}\} + \frac{1}{2}[\gamma^\mu, \not{p}]$

$$= -p^\mu - 2i S^{\mu\nu} p_\nu$$

$$\not{p}' \gamma^\mu = -p'^\mu + 2i S^{\mu\nu} p'_\nu$$

so $$\boxed{\bar{u}' \gamma^\mu u = \frac{1}{2m} \bar{u}' \left[(p+p')^\mu - 2i S^{\mu\nu} (p'-p)_\nu \right] u}$$

Gordon identity

~~Finally~~ Take $p \to (m, \vec{0})$ $\quad \vec{p}' \to (m, \vec{0})$ $\quad$ keep $q^\mu = p'^\mu - p^\mu$ to linear order

Recall: $u_s(\vec{0}) = \sqrt{m} \begin{pmatrix} \chi_s \\ \chi_s \end{pmatrix}$ where $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

in basis $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

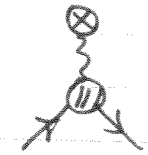
and $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -i \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} \Rightarrow \bar{u}'(\vec{0}) S^{0i} u(\vec{0}) = 0$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{-i}{4} \begin{pmatrix} [\sigma^i, \sigma^j] & \\ & [\sigma^i, \sigma^j] \end{pmatrix} = \frac{1}{2} \varepsilon_{ijk} \underbrace{\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix}}_{\equiv \Sigma_k}$$

so at tree level, as $q \to 0$ (to linear order in q):

$$F^\mu(q) \to \frac{e}{2m} \bar{u}'(\vec{0}) \left[2m \delta^\mu_0 - 2i \delta^\mu_i S^{ij} q_j \right] u(\vec{0})$$

$$= \frac{e}{2m} \bar{u}'(0) \left[2m \delta^\mu_0 - i \delta^\mu_i \varepsilon_{ijk} \Sigma_k q_j \right] u(\vec{0}) + \mathcal{O}(e^3) + \mathcal{O}(q^2)$$

so  at tree level gives

$$\bar{u}'(\vec{0})\left[eA_0 - \frac{e}{2m} i\varepsilon_{ijk} A^i q_j \Sigma_k\right] u(\vec{0})$$

$-i\varepsilon_{ijk}A^i q_j$ = Fourier transform of $(\vec{\nabla}\times\vec{A})_k = \vec{B}_k$

so get $\bar{u}'(0)\left[-eA^0 + \frac{e}{2m}\vec{\Sigma}\cdot\vec{B}\right]u(\vec{0})$

$$H = eA^0 - \frac{e}{2m}\vec{\Sigma}\cdot\vec{B} \qquad \vec{\Sigma} = \begin{pmatrix}\vec{\sigma} & \\ & \vec{\sigma}\end{pmatrix} = 2\vec{S}$$

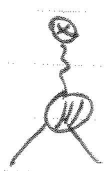
electron spin

$$= eA^0 - \vec{\mu}\cdot\vec{B}$$

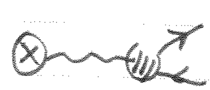
so $\vec{\mu} \equiv g\mu_B\vec{S}$, μ_B = Bohr magneton = $\frac{e}{2m}$
$g \equiv$ gyromagnetic factor

$\Rightarrow g = 2$ Dirac's triumph, justifies Pauli Hamiltonian devised to explain Stern-Gerlach experiment

Next: higher order contributions to g

iA =  = $i A^{cl}_{\mu} F^{\mu}$

general form of F^{μ} ?

Turn on side  photon turns into e^+e^- pair

γ: $J^{PC} = 1^{--}$ QED conserves angular momentum, parity and charge conjugation symmetries

$\therefore$ e^+e^- state must be $J^{PC} = 1^{--}$

Spin of e^+e^- equals $S=0$ or $S=1$. $\Rightarrow$ $l = 0, 1$ or 2 $(J=1)$

	l	S	P	C	
$J=1$:	0	1	-1	-1	
	1	0	$+1$	-1	X
	1	1	$+1$	-1	X
	2	1	-1	-1	

e^+e^- has intrinsic parity (-1), times $(-1)^l$ from orbit.

e^+e^- has intrinsic $C=-1$

- can only have $l=0$, or $l=2$
- only one Lorentz structure for each $\Rightarrow$ 2 independent form factors ... like tree level structure

$$F^\mu \equiv e\,\bar{u}'\left[\gamma^\mu F_1(q^2) - \frac{i S^{\mu\nu}}{m} q_\nu F_2(q^2)\right] u$$

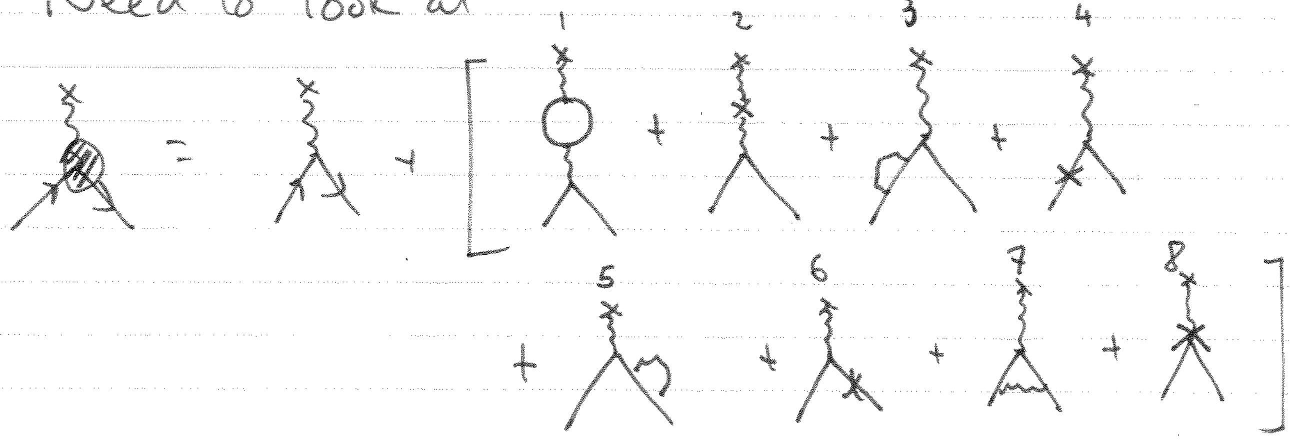
$q^2 \to 0$ must have $F_1(0) \to 1$ — says electron charge $= e$ at small momentum transfer

while $g = 2(\underbrace{1}_{\text{Dirac}} + \underbrace{F_2(0)}_{\text{Schwinger}})$

(Recall: at tree level there is no F_2 term: Dirac's $g=2$ comes from $F_1(q^2)$ term with $F_1(0)=1$)

So: We want to compute first correction to $(g-2) = 2F_2(0)$, to $\mathcal{O}(\alpha)$

Need to look at

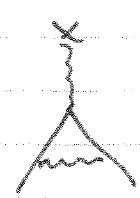


$+\mathcal{O}(e^5)$

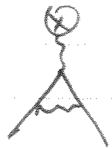
However: Note that the 1st six diagrams only contribute to $F_1(q^2)$... since Dirac's result $g=2$ used $F_1(0)=1$ which is true to all orders in perturbation theory (it is the renormalization condition: electron charge at $q^2=0$ equals e!)

Furthermore diagram 8 is a counterterm for $\bar\Psi \not{A} \Psi$ and can also only contribute to $F_1(q^2)$ (No renormalization of a $\bar\Psi S^{\mu\nu}F_{\mu\nu}\Psi$ operator in $\mathcal{L}$ since that is dim 5)

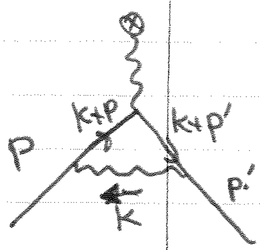
$\Rightarrow$ only contribution to $F_2(0)$ at $\mathcal{O}(e^2)$ comes from finite part of:



Since  $= i \not{A} \equiv i A^{cl}_\mu F^\mu$

$F^\mu \simeq$  $= \underbrace{-i\,(ie)^3\,(-i)^3}_{=-ie^3} \int \frac{d^4k}{(2\pi)^4}$

$\times \frac{1}{k^2 - i\epsilon} \; \frac{1}{(p+k)^2 + m^2 - i\epsilon} \; \frac{1}{(p'+k)^2 + m^2 - i\epsilon}$



$\times \; \bar{u}' \gamma_\lambda (-\not{p}' - \not{k} + m) \gamma^\mu (-\not{p} - \not{k} + m) \gamma^\lambda \, u$

use: $(-\not{p} + m)\gamma^\lambda u = \left[\gamma^\lambda (\not{p} + m) - \{\gamma^\lambda, \not{p}\}\right] u$

$= 2p^\lambda u \qquad \leftarrow (\not{p} + m)u = 0.$

$\bar{u}' \gamma_\lambda (-\not{p}' + m) = 2\bar{u} p'_\lambda \qquad \bar{u}'(\not{p}' + m) = 0$

$\therefore$ if $F^\mu = -ie^3 \int \frac{d^4k}{(2\pi)^4} \frac{N}{D}$

then · $N = \bar{u}'(2p'_\lambda - \gamma_\lambda \not{k}) \gamma^\mu (2p^\lambda - \not{k}\gamma^\lambda)\, u$

· D has no γ matrix structure

· We are looking for contribution to $F_2(q^2) S^{\mu\nu} q_\nu$ to $O(q)$

So: drop $p' \cdot p = -\frac{1}{2}(p' - p)^2 - m^2 = -\frac{1}{2} q^2 - m^2$ ← No $O(q)$ piece

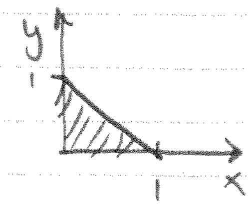
so $N \simeq \bar{u}'\left[-2\not{p}\not{k}\gamma^\mu - 2\gamma^\mu \not{k}\not{p}' + \gamma_\lambda \not{k}\gamma^\mu \not{k}\gamma^\lambda\right] u$

use identity: $\gamma_\lambda \not{k} \gamma^\mu \not{k} \gamma^\lambda = 2\not{k}\gamma^\mu \not{k}$

⑧

so $$\boxed{N \simeq \bar{u}' \left[-2 \not{p} \not{k} \gamma^\mu - 2 \gamma^\mu \not{k} \not{p}' + 2 \not{k} \gamma^\mu \not{k} \right] u}$$

Denominator: $$\frac{1}{abc} = 2 \int_\Delta dx\, dy \frac{1}{[ax + by + c(1-x-y)]^3}$$

$\Delta =$ 

$$\frac{1}{D} = \underbrace{\frac{1}{k^2 - i\epsilon}}_{c} \underbrace{\frac{1}{(p+k)^2 + m^2 - i\epsilon}}_{a} \underbrace{\frac{1}{(p'+k)^2 + m^2 - i\epsilon}}_{b}$$

$$= 2 \int_\Delta dx\, dy \frac{1}{[k^2 + 2p\cdot k\, x + 2p'\cdot k\, y - i\epsilon]^3}$$

~~used~~ used: $p^2 = p'^2 = -m^2$

Shift: $k' = k + px + p'y$

$$\frac{1}{D} = 2 \int_\Delta dx\, dy \frac{1}{[k'^2 - (px + p'y)^2]^3}$$

$$= 2 \int_\Delta dx\, dy \frac{1}{[k'^2 + m^2 x^2 + m^2 y^2 - 2p\cdot p'\, xy]^3}$$

as we have seen: $p\cdot p' = -\frac{1}{2} q^2 - \frac{1}{2} m^2 = -\frac{1}{2} m^2 + \mathcal{O}(q^2)$

drop: we want F^μ to $\mathcal{O}(q)$

$$\boxed{\frac{1}{D} \simeq 2 \int_\Delta dx\, dy \frac{1}{[k'^2 + m^2 (x+y)^2]^3} + \mathcal{O}(q^2)}$$

back to numerator: • $k = k' - (px + p'y)$

• drop odd powers of k', which integrate to zero

$$N \approx \bar{u}' \Big[-2\not{p}\big(\not{k}' - (\not{p}x + \not{p}'y)\big)\gamma^\mu - 2\gamma^\mu\big(\not{k}' - (\not{p}x + \not{p}'y)\big)\not{p}'$$

$$+2\big(\not{k}' - (\not{p}x + \not{p}'y)\big)\gamma^\mu\big(\not{k}' - (\not{p}x + \not{p}'y)\big)\Big] u$$

$$\simeq \bar{u}' \Big[+2\not{p}(\not{p}x + \not{p}'y)\gamma^\mu + 2\gamma^\mu(\not{p}x + \not{p}'y)\not{p}'$$

$$+ 2\not{k}'\gamma^\mu\not{k}' + 2(\not{p}x + \not{p}'y)\gamma^\mu(\not{p}x + \not{p}'y)\Big] u$$

• Can replace $k'^\mu k'^\nu \to \frac{1}{4} k'^2 g^{\mu\nu}$

so $\not{k}'\gamma^\mu\not{k}' \to \frac{1}{4} k'^2 \gamma_\alpha \gamma^\mu \gamma^\alpha = \frac{1}{2} (k')^2 \gamma^\mu$

... only contributes to F_1, so drop

• $\not{p}^2 = -p^2 = m^2$ so $\not{p}^2\gamma^\mu = m^2\gamma^\mu \to$ only contributes to F_1 ... drop

• $\gamma^\mu \not{p}'^2 \leftrightarrow$ drop, same reason

So $N \simeq 2\bar{u}' \big[\not{p}\not{p}'\gamma^\mu y + \gamma^\mu \not{p}\not{p}' x + (\not{p}x + \not{p}'y)\gamma^\mu(\not{p}x + \not{p}'y)\big] u$

Next: $\bar{u}\not{p}\not{p}'\gamma^\mu = \bar{u}'(\{\not{p}, \not{p}'\} - \not{p}'\not{p})\gamma^\mu$

$$= \bar{u}'(-2p\cdot p' + m\not{p})\gamma^\mu$$

F_1 contribution, drop.

so $\quad N \simeq 2\bar{u}'\left[m y \not{p} \gamma^\mu + m x \gamma^\mu \not{p}' + (\not{p} x + \not{p}' y)\gamma^\mu(\not{p} x + \not{p}' y)\right] u$

Next: use $p' = p + q$, $p = p' - q$, $\not{p} u = -m u$, $\bar{u}\not{p}' = -m\bar{u}'$

-- and keep dropping F_1 terms:

$$N = 2\bar{u}'\Big[-m y \not{q} \gamma^\mu + m x \gamma^\mu \not{q} + \big(-m(x+y) - \not{q} x\big)\gamma^\mu \big(-m(x+y) + \not{q} y\big)\Big] u$$

$$\simeq 2\bar{u}'\left[-m y \not{q}\gamma^\mu + m x \gamma^\mu \not{q} + m x (x+y) \not{q} \gamma^\mu - m y (x+y) \gamma^\mu \not{q}\right] u$$

(dropped $\mathcal{O}(q^2)$ + F_1 terms)

Since D = symmetric in x, y as is region Δ,

symmetrize N in $x + y$:

$$N \simeq \bar{u}'\left[m(x+y)[\gamma^\mu, \not{q}] - m(x+y)^2[\gamma^\mu, \not{q}]\right] u$$

$$= m(x+y)(1-(x+y))\, \bar{u}' \underbrace{[\gamma^\mu, \not{q}]}_{-4 i q_\nu S^{\mu\nu}} u$$

$$F^\mu = e\bar{u}'\left[\gamma^\mu F_1 - \frac{i S^{\mu\nu}}{m} q_\nu F_2\right] u = \cancel{-ie^3\int \frac{d^4k}{(2\pi)^4}\frac{N}{D}} = -ie^3\, 2\int_\Delta dx\, dy \int \frac{d^4k}{(2\pi)^4} \frac{N}{D}$$

So $F_2(0) = \frac{m}{e}\cdot(-ie^3)\cdot 2\int_\Delta dx\,dy\cdot\int\frac{d^4k}{(2\pi)^4}\cdot\frac{4\,m\,(x+y)(1-(x+y))}{\left[k^2+m^2(x+y)^2\right]^3}$

$= \cancel{8m^2} = 8m^2(-ie^2)\int dx\,dy\int\frac{d^4k}{(2\pi)^4}\,\frac{(x+y)(1-(x+y))}{\left[k^2+m^2(x+y)^2\right]^3}$

$d^4k \to i\,d^4k^E$

4D solid angle is $2\pi^2$

$$F_2(0) = 8m^2e^2\int_\Delta dx\,dy\cdot\frac{2\pi^2}{(2\pi)^4}\underbrace{\int_0^\infty k^3\,dk\,\left[k^2+m^2(x+y)^2\right]^{-2}}_{\frac{1}{4m^2(x+y)^2}}\left[(x+y)(1-(x+y))\right]$$

$\cancel{\frac{m^2e^2}{\pi^2}}$

$$= \frac{e^2}{4\pi^2}\int_\Delta dx\,dy\,\underbrace{\frac{(x+y)[1-(x+y)]}{(x+y)^2}}$$

$$= \frac{\alpha}{\pi}\int_0^1 dx\int_0^{1-x}dy\left(\frac{1}{(x+y)}-1\right)$$

$$= \frac{\alpha}{\pi}\int_0^1 dx\,\left[\ln\frac{1}{x}-(1-x)\right]$$

$$= \frac{\alpha}{\pi}\left[-x\ln x + x - \left(x-\frac{x^2}{2}\right)\right]_0^1$$

$$= \frac{\alpha}{2\pi}$$

$2\mu_B$

$$\boxed{\text{So } F_2(0) = \frac{\alpha}{2\pi} \qquad \mu = \frac{e}{m}\overbrace{\left(1+\frac{\alpha}{2\pi}\right)}^{g/2} \qquad g-2 = \frac{\alpha}{\pi}}$$

so $\quad g-2 = \frac{\alpha}{\pi} + \mathcal{O}(\alpha^2)$

Note: Independent of m, so should be the same for the $\mu, \tau \ldots$

$a^{th} = \frac{1}{2}(g-2) = \frac{\alpha}{2\pi} = 0.001164$

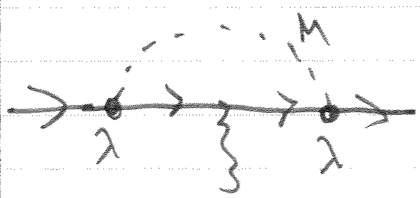
$a^{e}_{exp} = \quad 0.00115965218091\ (26)$

$a^{\mu}_{exp} = \quad 0.0011659209\ (6)$

discepancy for 1-loop result is few $\times 10^{-6}$

$$\left(\frac{\alpha}{2\pi}\right)^2 \approx 10^{-6} \quad \checkmark$$

μ is more interesting than e. Why? Suppose a heavy boson of mass M couples to the electron or muon:



This will contribute to $g-2$

Effective operator is dim 5:

$\bar{\Psi} F_{\mu\nu} S^{\mu\nu} \Psi$

- couples LH to RH fermions $\Rightarrow$ there must be a chirality flip in diagram $\propto m$

$\therefore$ operator goes as:

$\cancel{g-2 \sim \frac{\lambda^2}{8\pi^4} \frac{m}{M^2}}$ $\quad \frac{\lambda^2}{8\pi^2} \frac{m}{M^2} \cdot e \cdot \bar{\Psi} S^{\mu\nu} F_{\mu\nu} \Psi$

$\Rightarrow$ factor out $\frac{e}{m} \Rightarrow \quad g-2 \sim \frac{\lambda^2}{8\pi^2} \frac{m^2}{M^2}$

since $m_\mu \sim 2000\, m_e$ (106 MeV vs .511 MeV)

the μ g-2 is $\sim 4\times10^6$ times more sensitive to new physics

$$= \left(\frac{m_\mu}{m_e}\right)^2$$

This more than makes up for the greater accuracy of the g-2 measurement of e^-.

$(g-2)_\mu$ is at the 10^{-10} level

$$\frac{\lambda^2}{8\pi^2}\frac{m_\mu^2}{M^2} \approx 10^{-10} \Rightarrow M \sim 10^{+5}\cdot m_\mu \cdot \sqrt{\frac{\lambda^2}{8\pi^2}}$$

$\Rightarrow$ for $\frac{\lambda^2}{8\pi^2} \sim 1 \Rightarrow$ sensitive to $M \sim 10$ TeV.

if $\frac{\lambda^2}{8\pi^2} \sim \frac{\alpha}{2\pi} \Rightarrow$ sensitive to $M \sim \frac{1}{3}$ TeV