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Anomalies

2.1 The $U(1)_A$ anomaly in 1+1 dimensions

One of the fascinating features of chiral symmetry is that sometimes it is not a symmetry of the quantum field theory even when it is a symmetry of the Lagrangian. In particular, Noether's theorem can be modified in a theory with an infinite number of degrees of freedom; the modification is called "an anomaly". Anomalies turn out to be very relevant both for phenomenology, and for the implementation of lattice field theory. The reason anomalies affect chiral symmetries is that regularization requires a cut-off on the infinite number of modes above some mass scale, while chiral symmetry is incompatible with fermion masses¹.

Anomalies can be seen in many different ways. I think the most physical is to look at what happens to the ground state of a theory with a single flavor of massless Dirac fermion in (1 + 1) dimensions in the presence of an electric field. Suppose one adiabatically turns on a constant positive electric field $E(t)$, then later turns it off; the equation of motion for the fermion is ² $\frac{dp}{dt} = eE(t)$ and the total change in momentum is

$$\Delta p = e \int E(t) dt . \quad (2.1)$$

Thus the momenta of both left- and right-moving modes increase; if one starts in the ground state of the theory with filled Dirac sea, after the electric field has turned off, both the right-moving and left-moving sea has shifted to the right as in Fig. 2.1. The the final state differs from the original by the creation of particle- antiparticle pairs: right moving particles and left moving antiparticles. Thus while there is a fermion current in the final state, fermion number has not changed. This is what one would expect from conservation of the $U(1)$ current:

$$\partial_\mu J^\mu = 0 , \quad (2.2)$$

However, recall that right-moving and left-moving particles have positive and negative chirality respectively; therefore the final state in Fig. 2.1 has net axial charge, even though the initial state did not. This is peculiar, since the coupling of the electromagnetic field in the Lagrangian does not violate chirality. We can quantify the effect: if

¹Dimensional regularization is not a loophole, since chiral symmetry cannot be analytically continued away from odd space dimensions.

²While in much of these lectures I will normalize gauge fields so that $D_\mu = \partial_\mu + iA_\mu$, in this section I need to put the gauge coupling back in. If you want to return to the nicer normalization, set the gauge coupling to unity, and put a $1/g^2$ factor in front of the gauge action.

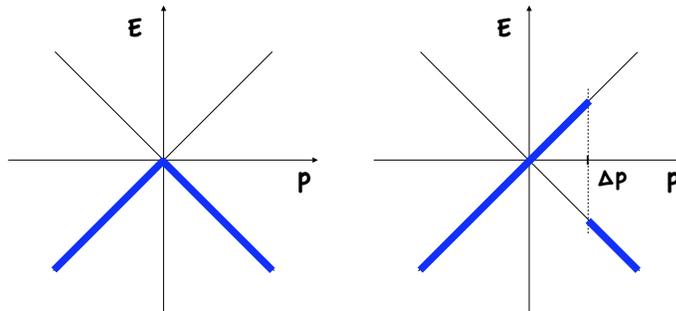


Fig. 2.1 On the left: the ground state for a theory of a single massless Dirac fermion in $(1 + 1)$ dimensions; on the right: the theory after application of an adiabatic electric field with all states shifted to the right by Δp , given in eqn. (2.1). Filled states are indicated by the heavier blue lines.

we place the system in a box of size L with periodic boundary conditions, momenta are quantized as $p_n = 2\pi n/L$. The change in axial charge is then

$$\Delta Q_A = 2 \frac{\Delta p}{2\pi/L} = \frac{e}{\pi} \int d^2x E(t) = \frac{e}{2\pi} \int d^2x \epsilon_{\mu\nu} F^{\mu\nu} , \quad (2.3)$$

where I expressed the electric field in terms of the field strength F , where $F^{01} = -F^{10} = E$. This can be converted into the local equation using $\Delta Q_A = \int d^2x \partial_\mu J_A^\mu$, a modification of eqn. (1.26):

$$\partial_\mu J_A^\mu = 2im\bar{\Psi}\Gamma\Psi + \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} , \quad (2.4)$$

where in the above equation I have included the classical violation due to a mass term as well. The second term is the axial anomaly in $1 + 1$ dimensions; it would vanish for a nonabelian gauge field, due to the trace over the gauge generator.

So how did an electric field end up violating chiral charge? Note that this analysis relied on the Dirac sea being infinitely deep. If there had been a finite number of negative energy states, then they would have shifted to higher momentum, but there would have been no change in the axial charge. With an infinite number of degrees of freedom, though, one can have a ‘‘Hilbert Hotel’’: the infinite hotel which can always accommodate another visitor, even when full, by moving each guest to the next room and thereby opening up a room for the newcomer. This should tell you that it will not be straightforward to represent chiral symmetry on the lattice: a lattice field theory approximates quantum field theory with a finite number of degrees of freedom — the lattice is a big hotel, but quite conventional. In such a hotel there can be no anomaly.

We can derive the anomaly in other ways, such as by computing the anomaly diagram Fig. 2.2, or by following Fujikawa (Fujikawa, 1979; Fujikawa, 1980) and carefully accounting for the Jacobian from the measure of the path integral when performing a chiral transformation. It is particularly instructive for our later discussion of lattice

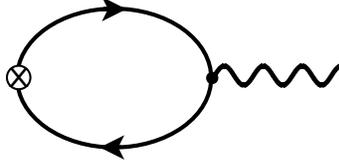


Fig. 2.2 The anomaly diagram in 1+1 dimensions, with one Pauli-Villars loop and an insertion of $2iM\bar{\Phi}\Gamma\Phi$ at the X .

fermions to compute the anomaly in perturbation theory using Pauli-Villars regulators of mass M . We replace our axial current by a regulated current

$$J_{A,\text{reg}}^\mu = \bar{\Psi}\gamma^\mu\Gamma\Psi + \bar{\Phi}\gamma^\mu\Gamma\Phi, \quad (2.5)$$

where Φ is our Pauli-Villars field; it follows then that

$$\partial_\mu J_{A,\text{reg}}^\mu = 2im\bar{\Psi}\Gamma\Psi + 2iM\bar{\Phi}\Gamma\Phi. \quad (2.6)$$

We are interested in matrix elements of $J_{A,\text{reg}}^\mu$ in a background gauge field between states without any Pauli-Villars particles, and so we need to evaluate $\langle 2iM\bar{\Phi}\Gamma\Phi \rangle$ in a background gauge field and take the limit $M \rightarrow \infty$ to see if $\partial_\mu J_{A,\text{reg}}^\mu$ picks up any anomalous contributions that do not decouple as we remove the cutoff.

To compute $\langle 2iM\bar{\Phi}\Gamma\Phi \rangle$ we need to consider all Feynman diagrams with a Pauli-Villars loop, and insertion of the $\bar{\Phi}\Gamma\Phi$ operator, and any number of external $U(1)$ gauge fields. By gauge invariance, a graph with n external photon lines will contribute n powers of the field strength tensor $F^{\mu\nu}$. For power counting, it is convenient that we normalize the gauge field so that the covariant derivative is $D_\mu = (\partial_\mu + iA_\mu)$; then the gauge field has mass dimension 1, and $F^{\mu\nu}$ has dimension 2. In $(1+1)$ dimensions $\langle 2iM\bar{\Phi}\Gamma\Phi \rangle$ has dimension 2, and so simple dimensional analysis implies that the graph with n photon lines must make a contribution proportional to $(F^{\mu\nu})^n/M^{2(n-1)}$. Therefore only the graph in Fig. 2.2 with one photon insertion can make a contribution that survives the $M \rightarrow \infty$ limit (the graph with zero photons vanishes). Calculation of this diagram yields the same result for the divergence of the regulated axial current as we found in eqn. (2.4).

Exercise 2.1 Compute the diagram in Fig. 2.2 using the conventional normalization of the gauge field $D_\mu = (\partial_\mu + ieA_\mu)$ and verify that $2iM\langle\bar{\Phi}\Gamma\Phi\rangle = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}$ when $M \rightarrow \infty$.

Note that in this description of the anomaly we (i) effectively rendered the number of degrees of freedom finite by introducing the regulator; (ii) the regulator explicitly broke the chiral symmetry; (iii) as the regulator was removed, the symmetry breaking effects of the regulator never decoupled, indicating that the anomaly arises when the two vertices in Fig. 2.2 sit at the same spacetime point. While we used a Pauli-Villars regulator here, the use of a lattice regulator will have qualitatively similar features, with the inverse lattice spacing playing the role of the Pauli-Villars mass,

and we can turn these observations around: A lattice theory will not correctly reproduce anomalous symmetry currents in the continuum limit, unless that symmetry is broken explicitly by the lattice regulator. This means we would be foolish to expect to construct a lattice theory with exact chiral symmetry. But can the lattice break chiral symmetry just enough to explain the anomaly, without losing the important consequences of chiral symmetry at long distances (such as protecting fermion masses from renormalization)?

2.2 Anomalies in 3+1 dimensions

2.2.1 The $U(1)_A$ anomaly

An analogous violation of the $U(1)_A$ current occurs in 3 + 1 dimensions as well ³. One might guess that the analogue of $\epsilon_{\mu\nu}F^{\mu\nu} = 2E$ in the anomalous divergence eqn. (2.4) would be the quantity $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 8\vec{E}\cdot\vec{B}$, which has the right dimensions and properties under parity and time reversal. So we should consider the behavior a massless Dirac fermion in (3 + 1) in parallel constant E and B fields. First turn on a B field pointing in the \hat{z} direction: this gives rise to Landau levels, with energy levels E_n characterized by non-negative integers n as well as spin in the \hat{z} direction S_z and momentum p_z , where

$$E_n^2 = p_z^2 + (2n + 1)eB - 2eBS_z . \quad (2.7)$$

The number density of modes per unit transverse area is defined to be g_n , which can be derived by computing the zero-point energy in Landau modes and requiring that it yields the free fermion result as $B \rightarrow 0$. We have $g_n \rightarrow p_\perp dp_\perp / (2\pi)$ with $[(2n + 1)eB - 2eBS_z] \rightarrow p_\perp^2$, implying that

$$g_n = eB/2\pi . \quad (2.8)$$

The dispersion relation looks like that of an infinite number of one-dimensional fermions of mass $m_{n,\pm}$, where

$$m_{n,\pm}^2 = (2n + 1)eB - 2eBS_z , \quad S_z = \pm\frac{1}{2} . \quad (2.9)$$

The state with $n = 0$ and $S_z = +\frac{1}{2}$ is distinguished by having $m_{n,+} = 0$; it behaves like a *massless* one-dimensional Dirac fermion (with transverse density of states g_0) moving along the \hat{z} axis with dispersion relation $E = |p_z|$. If we now turn on an electric field also pointing along the \hat{z} direction we know what to expect from our analysis in 1 + 1 dimensions: we find an anomalous divergence of the axial current equal to

$$g_0 eE/\pi = e^2 EB/2\pi^2 = \left(\frac{e^2}{16\pi^2} \right) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} . \quad (2.10)$$

If we include an ordinary mass term in the 3 + 1 dimensional theory, then we get

³Part of the content of this section comes directly from John Preskill's class notes on the strong interactions, available at his web page: <http://www.theory.caltech.edu/~preskill/notes.html>.

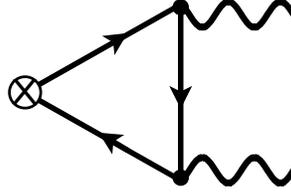


Fig. 2.3 The $U(1)_A$ anomaly diagram in 3+1 dimensions, with one Pauli-Villars loop and an insertion of $2iM\bar{\Phi}\Gamma\Phi$.

$$\partial_\mu J_A^\mu = 2im\bar{\Psi}\Gamma\Psi + \left(\frac{e^2}{16\pi^2}\right) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} . \quad (2.11)$$

One can derive this result by computing $\langle M\bar{\Phi}i\Gamma\Phi \rangle$ for a Pauli-Villars regulator as in the 1 + 1 dimensional example; now the relevant graph is the triangle diagram of Fig. 2.3.

If the external fields are nonabelian, the analogue of eqn. (2.11) is

$$\partial_\mu J_A^\mu = 2im\bar{\Psi}\Gamma\Psi + \left(\frac{g^2}{16\pi^2}\right) \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_b^{\rho\sigma} \text{Tr } T_a T_b . \quad (2.12)$$

If the fermions transform in the defining representation of $SU(N)$, it is conventional to normalize the coupling g so that $\text{Tr } T_a T_b = \frac{1}{2}\delta_{ab}$. This is still called an ‘‘Abelian anomaly’’, since J_A^μ generates a $U(1)$ symmetry.

2.2.2 Anomalies in Euclidian spacetime

Continuing to Euclidian spacetime by means of eqns. (1.11)-(1.15) changes the anomaly equations simply by eliminating the factor of i from in front of the fermion mass:

$$2d : \quad \partial_\mu J_A^\mu = 2m\bar{\Psi}\Gamma\Psi + \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} \quad (2.13)$$

$$4d : \quad \partial_\mu J_A^\mu = 2m\bar{\Psi}\Gamma\Psi + \left(\frac{g^2}{16\pi^2}\right) \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_b^{\rho\sigma} \text{Tr } T_a T_b . \quad (2.14)$$

2.2.3 The index theorem in four dimensions

For nonabelian gauge theories the quantity on the far right of eqn. (2.14) is a topological charge density, with

$$\nu = \frac{g^2}{64\pi^2} \int d^4x_E \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_a^{\rho\sigma} \quad (2.15)$$

being the winding number associated with $\pi_3(G)$, the homotopy group of maps of S_3 (spacetime infinity) into the gauge group G .

Consider then continuing the anomaly equation eqn. (2.12) to Euclidian space and integrating over spacetime its vacuum expectation value in a background gauge field (assuming the fermions to be in the N -dimensional representation of $SU(N)$) so that

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$\text{Tr } T_a T_b = \frac{1}{2} \delta_{ab}$). The integral of $\partial_\mu \langle J_A^\mu \rangle$ vanishes because it is a pure divergence, so we get

$$\int d^4 x_E m \langle \bar{\Psi} \Gamma \Psi \rangle = -\nu . \quad (2.16)$$

The matrix element above on the right equals

$$\int [d\Psi][d\bar{\Psi}] e^{-S_E} (m \bar{\Psi} \Gamma \Psi) / \int [d\Psi][d\bar{\Psi}] e^{-S_E} . \quad (2.17)$$

where $S_E = \bar{\Psi} (\not{D}_E + m) \Psi$. We can expand Ψ and $\bar{\Psi}$ in terms of eigenstates of the anti-hermitian operator \not{D}_E , where

$$\not{D}_E \psi_n = i\lambda_n \psi_n , \quad \int d^4 x_E \psi_m^\dagger \psi_n = \delta_{mn} , \quad (2.18)$$

with

$$\Psi = \sum c_n \psi_n , \quad \bar{\Psi} = \sum \bar{c}_n \psi_n^\dagger . \quad (2.19)$$

Then

$$\begin{aligned} \int d^4 x_E m \langle \bar{\Psi} \Gamma \Psi \rangle &= \left(\sum_n \int d^4 x_E m \psi_n^\dagger \Gamma \psi_n \prod_{k \neq n} (i\lambda_k + m) \right) / \prod_k (i\lambda_k + m) \\ &= m \sum_n \int d^4 x_E \psi_n^\dagger \Gamma \psi_n / (i\lambda_n + m) . \end{aligned} \quad (2.20)$$

Recall that $\{\Gamma, \not{D}\} = 0$; thus

$$\not{D} \psi_n = i\lambda_n \psi_n \quad \text{implies} \quad \not{D} (\Gamma \psi_n) = -i\lambda_n (\Gamma \psi_n) . \quad (2.21)$$

Thus for $\lambda_n \neq 0$, the eigenstates ψ_n and $(\Gamma \psi_n)$ must be orthogonal to each other (they are both eigenstates of \not{D} with different eigenvalues), and so $\psi_n^\dagger \Gamma \psi_n$ vanishes for $\lambda_n \neq 0$ and does not contribute to the sum in eqn. (2.20). In contrast, modes with $\lambda_n = 0$ can simultaneously be eigenstates of \not{D} and of Γ ; let n_+ , n_- be the number of RH and LH zeromodes respectively. The last integral in then just equals $(n_+ - n_-) = (n_R - n_L)$, and combining with eqn. (2.16) we arrive at the index equation

$$n_- - n_+ = \nu , \quad (2.22)$$

which states that the difference in the number of LH and RH zeromode solutions to the Euclidian Dirac equation in a background gauge field equals the winding number of the gauge field. With N_f flavors, the index equation is trivially modified to read

$$n_- - n_+ = N_f \nu . \quad (2.23)$$

This link between eigenvalues of the Dirac operator and the topological winding number of the gauge field provides a precise definition for the topological winding number of a gauge field on the lattice (where there is no topology) — provided we have a definition of a lattice Dirac operator which exhibits exact zeromodes. We will see that the overlap operator is such an operator.

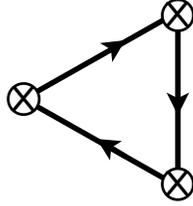


Fig. 2.4 Anomalous three-point function of three currents.

2.2.4 More general anomalies

Even more generally, one can consider the 3-point correlation function of three arbitrary currents as in Fig. 2.4,

$$\langle J_a^\alpha(k) J_b^\beta(p) J_c^\gamma(q) \rangle, \quad (2.24)$$

and show that the divergence with respect to any of the indices is proportional to a particular group theory factor

$$k_\mu \langle J_a^\mu(k) J_b^\alpha(p) J_c^\beta(q) \rangle \propto \text{Tr } Q_a \{Q_b, Q_c\} \Big|_{R-L} \epsilon^{\alpha\beta\rho\sigma} k_\rho k_\sigma, \quad (2.25)$$

where the Q s are the generators associated with the three currents in the fermion representation, the symmetrized trace being computed as the difference between the contributions from RH and LH fermions in the theory. The anomaly \mathcal{A} for the fermion representation is defined by the group theory factor

$$\text{Tr } (Q_a \{Q_b, Q_c\}) \Big|_{R-L} \equiv \mathcal{A} d_{abc}, \quad (2.26)$$

with d_{abc} being the totally symmetric invariant tensor of the symmetry group. For a simple group G (implying G is not $U(1)$ and has no factor subgroups), d_{abc} is only nonzero for $G = SU(N)$ with $N \geq 3$; even in the case of $SU(N)$, d_{abc} will vanish for real irreducible representations (for which $Q_a = -Q_a^*$), or for judiciously chosen reducible complex representations, such as $\bar{5} \oplus 10$ in $SU(5)$. For a semi-simple group $G_1 \times G_2$ (where G_1 and G_2 are themselves simple) there are no mixed anomalies since the generators are all traceless, implying that if $Q \in G_1$ and $Q \in G_2$ then $\text{Tr } (Q_a \{Q_b, Q_c\}) \propto \text{Tr } Q_a = 0$. When considering groups with $U(1)$ factors there can be nonzero mixed anomalies of the form $U(1)G^2$ and $U(1)^3$ where G is simple; the $U(1)^3$ anomalies can involve different $U(1)$ groups. With a little group theory it is not difficult to compute the contribution to the anomaly of any particular group representation.

If a current with an anomalous divergence is gauged, then the theory does not make sense. That is because the divergencelessness of the current is required for the unphysical modes in the gauge field A_μ to decouple; if they do not decouple, their propagator has a piece that goes as $k_\mu k_\nu / k^2$ which does not fall off at large momentum, and the theory is not renormalizable.

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When global $U(1)$ currents have anomalous divergences, that is interesting. We have seen that the $U(1)_A$ current is anomalous, which explains the η' mass; the divergence of the axial isospin current explains the decay $\pi^0 \rightarrow \gamma\gamma$; the anomalous divergence of the baryon number current in background $SU(2)$ in the Standard Model predicts baryon violation in the early universe and the possibility of weak-scale baryogenesis.

Exercise 2.2 Verify that all the gauge currents are anomaly-free in the standard model with the representation in eqn. (1.45). The only possible G^3 anomalies are for $G = SU(3)$ or $G = U(1)$; for the $SU(3)^3$ anomaly use the fact that a LH Weyl fermion contributes $+1$ to \mathcal{A} if it transforms as a 3 of $SU(3)$, and contributes -1 to \mathcal{A} if it is a $\bar{3}$. There are two mixed anomalies to check as well: $U(1)SU(2)^2$ and $U(1)SU(3)^2$.

This apparently miraculous cancellation is suggestive that each family of fermions may be unified into a spinor of $SO(10)$, since the vanishing of anomalies which happens automatically in $SO(10)$ is of course maintained when the symmetry is broken to a smaller subgroup, such as the Standard Model.

Exercise 2.3 Show that the global B (baryon number) and L (lepton number) currents are anomalous in the Standard Model eqn. (1.45), but that $B - L$ is not.

2.3 Strongly coupled chiral gauge theories

Strongly coupled chiral gauge theories are particularly intriguing, since they can contain light composite fermions, which could possibly describe the quarks and leptons we see. A nice toy example of a strongly coupled chiral gauge theory is $SU(5)$ with LH fermions

$$\psi = \bar{5}, \quad \chi = 10. \quad (2.27)$$

It so happens that the ψ and the χ contribute with opposite signs to the $(SU(5))^3$ anomaly \mathcal{A} in eqn. (2.26), so this seems to be a well defined gauge theory. Furthermore, the $SU(5)$ gauge interactions are asymptotically free, meaning that interactions becomes strong at long distances. One might therefore expect the theory to confine as QCD does. However, unlike QCD, there are no gauge invariant fermion bilinear condensates which could form, and which in QCD are responsible for baryon masses. That being the case, might there be any massless composite fermions in the spectrum of this theory? 't Hooft came up with a nice general argument involving global anomalies that suggests there will be.

In principle there are two global $U(1)$ chiral symmetries in this theory corresponding to independent phase rotations for ψ and χ ; however both of these rotations have global $\times SU(5)^2$ anomalies, similar to the global $\times SU(3)^2$ of the $U(1)_A$ current in QCD. This anomaly can only break one linear combination of the two $U(1)$ symmetries, and one can choose the orthogonal linear combination which is anomaly-free. With a little group theory you can show that the anomaly-free global $U(1)$ symmetry corresponds to assigning charges

$$\psi = \bar{5}_3, \quad \chi = 10_{-1}, \quad (2.28)$$

where the subscript gives the global $U(1)$ charge. This theory has a nontrivial global $U(1)^3$ anomaly, $\mathcal{A} = 5 \times (3)^3 + 10 \times (-1)^3 = 125$. 't Hooft's argument is that