I. NJL MODEL LECTURE NOTES - PHYSICS 571 - WINTER 2014

A. The three dimensional, large-N NJL model

Nambu and Jona-Lasinio invented a model for studying relativistic fermions with a strong attraction, modeled after the BCS theory of superconductivity. Here I will describe a version that is simpler to analyze and is better defined than the original model. The starting point is a theory of N massless Dirac fermions with an attractive 4-fermion interaction, and the theory is formulated in Euclidian spacetime (repeated a, b indices are summed from $1, \dots, N$):

$$\mathcal{L} = \left[\overline{\psi}_a \partial\!\!\!/ \psi_a - \frac{g}{2N}\left[(\overline{\psi}_a\psi_a)^2 + (\overline{\psi}_a i\gamma_5\psi_a)^2\right]\right] = \left[\overline{\psi}_{La}\partial\!\!\!/\psi_{La} + \overline{\psi}_{Ra}\partial\!\!\!/\psi_{Ra} - \frac{g}{N}(\overline{\psi}_{La}\psi_{Ra})(\overline{\psi}_{Rb}\psi_{Lb})\right] . \tag{1}$$

In Euclidian spacetime the metric is the unit matrix (so we can write every index as a lower index) and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. In this expression we put an explicit factor of $1/N$ in coupling; later we will be taking $N \to \infty$ with g fixed, which will simplify the theory. The partition function is then

$$Z = \mathcal{N}\int D\psi\, D\overline{\psi}\, e^{-S} , \qquad S = \int \mathcal{L} . \tag{2}$$

Given the second form for $\mathcal{L}$ in eq. (1), it is evident that the Lagrangian has a $U(N) \times U(1)_A$ symmetry. Under the $U(N)$, $\psi_a \to U_{ab}\psi_b$, where U is an $N \times N$ unitary matrix which mixes up the N "flavors". It acts the same way on ψ_{La} and ψ_{Ra}, so it is called a vector symmetry. The $U(1)_A$ symmetry acts as $\psi_a \to \exp[i\alpha\gamma_5]\psi_a$; it rotates ψ_{Ra} and ψ_{La} with opposite phases, but does not mix up flavors, and is called an axial symmetry. A common mass term for all the fermions would explicitly break the $U(1)_A$ symmetry but not the $U(N)$ symmetry.

Now what we are going to do is take this 4d theory and simply erase one of the coordinates (this is like forcing the fermions to lie in a spatial plane). We will keep all the gamma matrices unchanged, we just remove all x_4 dependence form the fields and eliminate $\partial_4\gamma_4$ from $\partial\!\!\!/$. The only practical effect this will have is that loop momenta will de integrated $d^3p/(2\pi)^3$ instead of $d^4p/(2\pi)^4$. The purpose for doing this is to make the renormalization of the theory simpler. However, it does also change the dimension of the coupling g: in 3d the Lagrange density must have mass dimension 3, so ψ has dimension mass^1 and g has dimension mass^{-1}.

B. Introducing the σ and π auxiliary fields

It is convenient to write the theory entirely in terms of fermion bilinears by introducing real auxiliary boson fields σ and π, using what we know about gaussian integrals. We write a new Lagrangian

$$\mathcal{L}' = \left(\frac{N}{2g}\left(\sigma^2 + \pi^2\right) + \sum_{a=1}^{N}\overline{\psi}_a\left[\partial\!\!\!/ + \sigma + i\pi\gamma_5\right]\psi_a\right) . \tag{3}$$

Note that the new fields σ and π do not have kinetic terms. Therefore it is easy to show that

$$Z = \mathcal{N}\int D\psi\, D\overline{\psi}\, e^{-S} = \mathcal{N}'\int D\psi\, D\overline{\psi}\int D\sigma\, D\pi\, e^{-S'} \tag{4}$$

by simply completing the square in the σ and π integrals and then performing the resulting gaussian integrals, absorbing an uninteresting overall constant into the normalization of the path integral. Note that the fermion part of $\mathcal{L}'$ consists of N fermions with identical interactions.

The $U(N) \times U(1)_A$ symmetry is still evident in $\mathcal{L}'$, where the fermions transform as before, while the complex field $\phi = (\sigma + i\pi)/\sqrt{2}$ is invariant under the $U(N)$ and transforms as $\phi \to \exp(-2i\alpha)\Phi$ under the $U(1)_A$.

In second form for Z one can compute the expectation values of σ and π and find

$$\langle\sigma\rangle = \frac{g}{N}\sum_a\langle\overline{\psi}_a\psi_a\rangle , \qquad \langle\pi\rangle = \frac{g}{N}\sum_a\langle\overline{\psi}_a i\gamma_5\psi_a\rangle . \tag{5}$$

What we are going to do now is integrate out the fermions, leaving us with just a theory of the bosons; if we find spontaneous symmetry breaking of the $U(1)_A$ symmetry in that theory then we have shown that the original theory has spontaneously broken the axial symmetry by causing the fermion bilinear $\overline{\psi}\psi$ to get a vacuum expectation value. Spontaneous symmetry breaking by fermions – pretty neat!

C. The scalar theory

We now integrate out the fermions, which we can do now that the action is bilinear in the fermion fields (giving us a determinant) to get the scalar theory (dropping the uninteresting normalization of Z):

$$Z = \int D\sigma\, D\pi\, e^{-S_\phi}\ , \qquad S_\phi = \int d^3x\, \mathcal{L}_\phi\ , \tag{6}$$

$$\mathcal{L}_\phi = N\left(\frac{1}{2g}\left(\sigma^2+\pi^2\right) - \mathrm{Tr}\,\ln\left[\partial\!\!\!/ + \sigma + i\pi\gamma_5\right]\right)\ . \tag{7}$$

We have factored out an overall factor of N since each of the N fermions gives an identical Tr ln term. That means that the Tr means a trace over Dirac indices and an integral over momentum (or spacetime), but does not involve and flavor sum.

Note that g has dimension 1/mass.

1. The vacuum

The first thing to do is to see whether $\langle\phi\rangle = 0$ or not in the vacuum. Since we do not expect the vacuum to correspond to a nonzero spatially dependent field (which would spontaneously break Lorentz invariance) we need to just make ϕ a constant, and then minimize the scalar potential ($= -\mathcal{L}$ for a constant field). Since there is a $U(1)_A$ symmetry, we can always rotate the phase of ϕ so that its vev is real and positive, namely $\langle\pi\rangle = 0$, $\langle\sigma\rangle = f > 0$. So we only need to compute the value of f for a given g.

Normally this would be very difficult to do, since the vacuum energy is the sum of connected vacuum diagrams, which we do not know how to compute and sum. But in the large N limit we can compute the energy in a $1/N$ expansion. The parameter N only appears in front of the Lagrangian; or as $N/\hbar$ in front of the aciton. In a previous problem set you showed that a graph with L loops scales as $\hbar^{L-1}$, so that means for this theory a graph with L loops scales as N^{1-L}, and that the leading contribution to the vacuum energy will therefore come from tree diagrams (with external legs given by $\langle\sigma\rangle$). This is equivalent to simply evaluating Z in a saddle point approximation, solving for the stationary point

$$\frac{\partial}{\partial\sigma}S_\phi\Big|_{\sigma=f,\,\langle\pi\rangle=0} = N\left(\frac{f}{g} - \mathrm{Tr}\,\frac{1}{\partial\!\!\!/ + f}\right) = 0\ . \tag{8}$$

The trace is just

$$\mathrm{Tr}\left(\frac{1}{\partial\!\!\!/ + f}\right) = \int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\,\frac{1}{ip\!\!\!/ + f} = 4\int\frac{d^3p}{(2\pi)^3}\frac{f}{p^2+f^2}\ . \tag{9}$$

where the Tr in the first term on the LHS of the equation is a trace over the differential operator, while the Tr in the second term is just over Dirac indices. So our vacuum equation is

$$f = 0 \qquad \text{or} \qquad \frac{1}{g} = 4\int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+f^2}\ . \tag{10}$$

The integral is linearly divergent, but that divergence can be absorbed by renormalizing g. For example we could use a cutoff, and set

$$\frac{1}{g} = 4\int^\Lambda\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+f^2} = \frac{4}{(2\pi)^3}4\pi\int_0^\Lambda\frac{1}{p^2+f^2}p^2dp = \frac{2\left(\Lambda - f\tan^{-1}\left(\frac{\Lambda}{f}\right)\right)}{\pi^2} \tag{11}$$

Taking $\Lambda \gg f$ then gives

$$\tan^{-1}\left(\frac{\Lambda}{f}\right) = \frac{\pi}{2} + O\left(\frac{f}{\Lambda}\right)\ , \tag{12}$$

so eq. (11) becomes

$$\frac{1}{g_c} - \frac{1}{g} = \frac{f}{\pi}\ , \qquad g_c \equiv \frac{\pi^2}{2\Lambda}\ . \tag{13}$$

We see that for a fixed nonzero $f > 0$ we need to tune $g \to g_c$ from above ($g > g_c$), and that if we send $\Lambda \to \infty$ with fixed f, $g_c \to 0$.

Is this the true minimum? What about the $f = 0$ solution in eq. (10)? To figure out what the true minimum is could compute the potential $V(\sigma)$ by integrating with respect to σ (f) the derivative of the potential which we have computed. Or, to at least see if the solution $f = 0$ is a local minimum or maximum, we just need to compute $V''(f)$ at $f = 0$. The $f = 0$ solution is at least locally stable if [1]

$$\left.\frac{\partial^2 S}{\partial \sigma^2}\right|_{\langle\sigma\rangle=\langle\pi\rangle=0} > 0\ . \tag{14}$$

This is easily computed:

$$\left.\frac{\partial^2 S}{\partial \sigma^2}\right|_{\langle\sigma\rangle=\langle\pi\rangle=0} = \frac{1}{g} + \mathrm{Tr}\,\frac{1}{\not\partial}\frac{1}{\not\partial} = \frac{1}{g} - 4\int^{\Lambda}\frac{d^3p}{(2\pi)^3}\frac{1}{p^2} = \frac{1}{g} - \frac{1}{g_c}\ . \tag{15}$$

So we see that the $f = 0$ solution is the minimum for weak coupling, $g < g_c$, while the $f \neq 0$ solution in eq. (13) is the solution for strong coupling, $g > g_c$.

2. Spontaneous symmetry breaking and a Goldstone boson

We have seen that for strong coupling $g > g_c$ we have $\langle\sigma\rangle = f > 0$. Note that this implies spontaneous symmetry breaking of the $U(1)_A$. Also: from eq. (7) $\langle\sigma\rangle = f$ implies that the fermions acquire a mass $M = f$. Pretty neat! But shouldn't there be a Goldstone boson then? Yes, the π field, since an infinitesimal $U(1)_A$ transformation is

$$\delta\phi = -2i\alpha\phi \qquad \longrightarrow \qquad \delta\pi = -2\alpha\sigma = -2\alpha f\ . \tag{16}$$

This shift by a constant is the hallmark of a Goldstone boson – it ensures that the field can only have derivative couplings (and hence no mass). However, to see that there is a Goldstone boson it is not enough to show that there is no mass for the pion – one also has to show that it is a real propagating particle...even though $\mathcal{L}_\phi$ did not contain the usual kinetic term for the π field. It is generated by the fermion loop, and the way to compute it is to expand the Tr ln to second order in the pion field, assuming it carries nonzero momentum k (in the vacuum $\langle\sigma\rangle = f$). This gives the inverse pion propagator

$$\begin{aligned} D_\pi(k) &= \frac{1}{g} - \int^{\Lambda}\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left[\gamma_5\frac{1}{i(\not p + \not k/2) + f}\gamma_5\frac{1}{i(\not p - \not k/2) + f}\right] \\ &= \frac{k\cot^{-1}(2f/k)}{2\pi}\ . \end{aligned} \tag{17}$$

The pion propagator is therefore

$$G_\pi(k) = \frac{2\pi}{k\cot^{-1}(2f/k)} \equiv \frac{Z_\pi(k)}{k^2}\ , \qquad Z_\pi(k) = \frac{2\pi k}{\cot^{-1}(2f/k)} = 4\pi f\left(1 + \frac{1}{12}\left(\frac{k}{f}\right)^2 + O(k^4/f^4)\right)\ . \tag{18}$$

so we see that the pion is indeed propagating and massless, with the pole in its propagator at $k^2 = 0$.

Of you perform a similar computation for the σ propagator you find a strange singularity at $k^2 = (2f)^2$: the sigma particle would have a mass of $2f$...except that it isn't a particle because it can fall apart into a $\psi - \bar\psi$ pair at rest, which has a mass of $2f$. Thus we have symmetry breaking, producing the required Goldstone boson, but not an analogue of the Higgs boson (which is what the σ would be it it was a stable particle.) A similar mechanism has been proposed (called technicolor) to explain electroweak symmetry breaking due to fermion condensation, in place of the standard Higgs theory, but the discovery of a stable Higgs boson would seem to rule that option out.

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[1] Note that the saddlepoint solution for $e^{-NS(x)}$ with $S(x) = +\frac{x^2}{2} + \frac{x^4}{4}$ is at $x = 0$, while for $S(x) = -\frac{x^2}{2} + \frac{x^4}{4}$, the saddlepoint is at $x = \pm 1 \implies$ the $x = 0$ solution requires $S''(0) > 0$.