Physics 572

Fujikawa Anomaly Paper Note

This is a note on Fujikawa's papers on deriving the anomaly from the path integral:

- **Phys. Rev. Lett. 42 (1979) 1195**
- **Phys. Rev. D22 (1980) 1499**

Consider the Euclidian path integral for a massive fermion interacting with a gauge field:

$$Z = \mathcal{N} \int d\bar{\psi} d\psi \, e^{-S} \ , \qquad S = \int d^4x \left(\bar{\psi} \left(\not{D} + m \right) \psi + \frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu}^2 \right) \tag{1}$$

where

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \ , \quad \{\gamma_5, \gamma_\mu\} = 0 \ , \quad \gamma_5^\dagger = \gamma_5 \ , \quad \mathrm{Tr}\, \gamma_5 \gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\tau = 4\epsilon_{\mu\nu\sigma\tau} \ . \tag{2}$$

and

$$D_\mu = \partial_\mu + iA_\mu \ , \qquad A_\mu = A_\mu^a T_a \ , \qquad \mathrm{Tr}\, T_a T_b = \frac{1}{2}\delta_{ab} \ . \tag{3}$$

The operator $\not{D}$ is anti Hermitian, and has orthonormal eigenfunctions $\phi_n(x)$ such that

$$\not{D}\phi_n = i\lambda_n \phi_n \ , \qquad \not{D}\gamma_5\phi_n = -i\lambda_n \gamma_5 \phi_n \ , \quad \lambda_n \in \mathrm{Reals} \ , \tag{4}$$

We can use these eigenstates as a basis for expanding ψ and $\bar{\psi}$ in the path integral:

$$\psi(x) = \sum_n a_n \phi_n(x) \ , \qquad \bar{\psi}(x) = \sum_n \phi_n^\dagger(x) \bar{b}_n \ , \tag{5}$$

where the a_n, $\bar{b}_n$ are Grassmann numbers. The path integral measure is then

$$d\bar{\psi} d\psi = \prod_{m,n} \bar{b}_m a_n \ . \tag{6}$$

Under a change of variables corresponding to an x-dependent chiral rotation:

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)\gamma_5} \psi(x) \ , \qquad \bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) e^{i\alpha(x)\gamma_5} \ . \tag{7}$$

We will want to derive the Ward identity following from $\frac{\delta Z}{\delta\alpha(x)}|_{\alpha=0} = 0$. The "classical" contribution comes from the change in the action,

$$S \to S + \int d^4x \ i\partial_\mu \alpha(x) \bar{\psi}\gamma_\mu\gamma_5\psi + 2im\alpha(x)\bar{\psi}\gamma_5\psi + O(\alpha^2) \ . \tag{8}$$

The "anomalous" part arises from the Jacobian that arises from the path integral measure, which we need to compute. We write

$$\psi' = \sum_n a_n' \phi_n = \sum_{m,n} a_m C_{mn} \phi_n \ , \qquad \bar{\psi}' = \sum_{m,n} \phi_m^\dagger C_{mn} \bar{b}_n \tag{9}$$

where

$$C_{mn} = \int d^4x \phi_m^\dagger e^{i\alpha\gamma_5}\phi_n. \tag{10}$$

Therefore since $da_m = \frac{\partial}{\partial a_m}$ we find

$$d\bar{\psi}' d\psi' = d\bar{\psi} d\psi \det C^{-2} = d\bar{\psi} d\psi e^{-2\mathrm{Tr}\,\ln C} \ . \tag{11}$$

To compute the variation we want, we need only compute the log to linear order in α,

$$-2\mathrm{Tr}\,\ln C \simeq -2i\mathrm{Tr}\,\alpha(x)\gamma_5 \equiv -2i\int d^4x \alpha(x)\mathcal{A}(x)\ , \qquad \mathcal{A}(x) = \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x). \tag{12}$$

To define $\mathcal{A}(x)$ well we need to regulate it. Fujikawa chooses to regulate the sum by inserting a factor of $e^{-\lambda_n^2/M^2}$, subsequently taking the $M \to \infty$ limit. This is a good choice of regulator because it is gauge invariant. Thus we have

$$\mathcal{A}(x) = \lim_{M\to\infty} \langle x|\gamma_5 e^{\not{D}^2/M^2}|x\rangle \ . \tag{13}$$

We will wish to expand this in powers of the gauge field, so we write

$$\not{D}^2 = \not{\partial}^2 + \left(\not{D}^2 - \not{\partial}^2\right) \tag{14}$$

where the last term is $O(A_\mu)$. Now use the Baker-Campbell-Hausdorff theorem

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+\ldots}\ , \tag{15}$$

setting

$$X = -\frac{\not{\partial}^2}{M^2} = -\frac{\partial^2}{M^2}\ , \qquad Y = \frac{\not{D}^2}{M^2} = \frac{D_\mu D_\mu + \frac{i}{2}F_{\mu\nu}\gamma_\mu\gamma_\nu}{M^2} \tag{16}$$

where I used $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $[D_\mu, D_\nu] = iF_{\mu\nu}$. Then we have

$$\begin{aligned} e^{\not{D}/M^2} &= e^{\frac{\partial^2}{M^2}} e^{X+Y+\frac{1}{2}[X,Y]+\ldots} \\ &= e^{\frac{\partial^2}{M^2}}\Bigg[1 + \frac{1}{M^2}\left(2iA\cdot\partial + i(\partial\cdot A) - A^2 + \frac{i}{2}F_{\mu\nu}\gamma_\mu\gamma_\nu\right) \\ &\quad + \frac{1}{M^4}\left(\frac{1}{2}\left(2iA\cdot\partial + i(\partial\cdot A) - A^2 + \frac{i}{2}F_{\mu\nu}\gamma_\mu\gamma_\nu\right)^2 + \left[-\partial^2, \not{D}^2\right]\right) + O\left(\frac{1}{M^6}\right)\Bigg] \end{aligned} \tag{17}$$

Note that

$$\langle x|e^{\frac{\partial^2}{M^2}}\mathcal{O}|x\rangle = \int \frac{d^4k}{(2\pi)^4}\int d^4y\,\langle x|e^{\frac{\partial^2}{M^2}}|k\rangle\langle k|y\rangle\langle y|\mathcal{O}|x\rangle = \int d^4y\,\langle y|\mathcal{O}|x\rangle \int \frac{d^4k}{(2\pi)^4} e^{-\frac{k^2}{M^2}} e^{ik(y-x)}\ . \tag{18}$$

Note that:

1. The integral over k has dimensions of $(\text{mass})^4$ and hence can only grow as fast as M^4 in the large M limit; therefore from the $\langle y|\mathcal{O}|x\rangle$ we need only consider terms falling off no faster than $1/M^4$;

2. To compute $\mathcal{A}(x)$ in eq. (13), we need only pick out from $\mathcal{O}$ terms which do not vanish when traced over Dirac indices with a power of γ_5.

These two considerations greatly simplify our work, and we get

$$\begin{aligned}\mathcal{A}(x) &= \lim_{M\to\infty} -\frac{1}{8M^4}\mathrm{Tr}\,\gamma_5\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\tau \int d^4y\,\langle y|\mathrm{Tr}\,F_{\mu\nu}F_{\sigma\tau}|x\rangle \int \frac{d^4k}{(2\pi)^4}e^{-\frac{k^2}{M^2}}e^{ik(y-x)} \\ &= \lim_{M\to\infty} -\frac{1}{2M^4}\epsilon_{\mu\nu\sigma\tau}\mathrm{Tr}\,F_{\mu\nu}(x)F_{\sigma\tau}(x)\int \frac{d^4k}{(2\pi)^4}e^{-\frac{k^2}{M^2}} \\ &= -\frac{1}{16\pi^2}\mathrm{Tr}\,F\tilde{F}(x) \end{aligned} \tag{19}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}F_{\sigma\tau}$. In going from the first to the second line above I used $\langle x|F^2|y\rangle = F^2(x)\delta^4(x-y)$. Note that some of the traces are over the entire Hilbert space, some are over only Dirac indices, some are only over gauge group indices...you should be able tell from context.

We can now compute the Ward identities; the derivative of the partition function with respect to $\alpha(x)$ must vanish, because we just did a change of integration variables. Using our expression for $\mathcal{A}(x)$ and eq. (8) we have

$$\begin{aligned} 0 = \left.\frac{\delta Z}{\delta\alpha(x)}\right|_{\alpha=0} &= \left.\frac{\delta}{\delta\alpha(x)}\mathcal{N}\int d\bar{\psi}d\psi e^{-S-i\int d^4x\alpha(x)\left[-\partial_\mu\bar{\psi}\gamma_\mu\gamma_5\psi+2m\bar{\psi}\gamma_5\psi+2\mathcal{A}(x)\right]}\right|_{\alpha=0} \\ &= \left\langle -\partial_\mu\bar{\psi}\gamma_\mu\gamma_5\psi + 2m\bar{\psi}\gamma_5\psi - \frac{1}{8\pi^2}\mathrm{Tr}\,F\tilde{F}(x)\right\rangle \end{aligned} \tag{20}$$

This gives the Euclidian version of the Ward identity,

$$\partial_\mu\langle\bar{\psi}\gamma_\mu\gamma_5\psi\rangle = \langle 2m\bar{\psi}\gamma_5\psi - \frac{1}{8\pi^2}\mathrm{Tr}\,F\tilde{F}(x)\rangle \tag{21}$$

It is interesting to consider the integral of the anomaly; on the one hand we have

$$\int d^4x\mathcal{A}(x) = -\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\,F\tilde{F}\ , \tag{22}$$

while on the other we have

$$\int d^4x\,\mathcal{A}(x) = \sum_n \int d^4x\,\phi_n^\dagger(x)\gamma_5\phi_n(x)\ . \tag{23}$$

From eq. (4) we see that the functions ϕ_n and $\gamma_5\phi_n$ correspond to eigenvalues $\pm i\lambda_n$, and so must be orthogonal to each other for $\lambda_n \neq 0$; thus only the eigenfunctions with $\lambda_n = 0$ can contribute to the sum. If $\not{D}\phi = 0$, then $[\not{D},\gamma_5]\phi = 0$ and we can take ϕ to be an eigenstate of γ_5. Thus we find

$$\int d^4x\,\mathcal{A}(x) = n_+ - n_- \equiv \nu\ , \tag{24}$$

where $n_\pm$ are the number of positive and negative chirality zeromodes of the Dirac operator. Equating eq. (22) and eq. (24) we get the interesting result

$$-\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\,F\tilde{F} = \nu\ , \tag{25}$$

which is an integer. Seeing an integer result leads us to consider the topology of gauge fields.