Problem set #4, Physics 571 **Due Thursday 2/11/2014**

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1. (a) Show that for any connected Feynman graph, $L = P - V + 1$, where L is the number of loop integrations, V is the number of vertices, and P is the number of propagators (internal lines).

   (b) The parameter $\hbar$ only enters quantum field theory through the phase factor in the path integral, $e^{iS/\hbar}$, where S is independent of $\hbar$. Show that a connected Feynman diagram with L loops is proportional to $\hbar^{L-1}$. Therefore the loop expansion is sometimes called the "semiclassical" expansion. Note that in the limit $\hbar \to 0$ one would expect the stationary phase approximation to work; why does this correspond to the classical equations of motion?

2. The Fermi theory for the weak interactions of leptons involves the contact interaction

$$\mathcal{L}_{\rm int} = -\frac{G_F}{\sqrt{2}} J^\nu J^\dagger_\nu \ , \qquad J^\nu = \bar\psi_e \gamma^\nu (1-\gamma_5)\psi_{\nu_e} + \bar\psi_\mu \gamma^\nu (1-\gamma_5)\psi_{\nu_\mu} \ , \tag{1}$$

where the subscripts e, ν_e, μ, ν_μ refer to the electron, the electron neutrino, the muon, and the muon neutrino, each of which has its own Dirac field. G_F is called the Fermi constant and has dimension of $1/\text{mass}^2$:

$$G_F = 1.17 \times 10^{-5}\ \text{GeV}^{-2} \ . \tag{2}$$

In this problem you are to compute the lifetime of the muon, which has a mass

$$m_\mu \equiv M = 106\ \text{MeV} \ , \tag{3}$$

making the simplifying assumptions that the outgoing particles are all massless,

$$m_e = m_{\nu_e} = m_{\nu_\mu} = 0 \ . \tag{4}$$

This is a decent approximation, since the heaviest of the decay products is the electron, and $m_e \simeq m_\mu/200$.

   (a) Draw the tree level diagram for the decay of the muon, $\mu^- \to e^- \nu_\mu \bar\nu_e$ where $\bar\nu_e$ is the anti-particle of the electron neutrino. Designate the momenta as $p_\mu \equiv p$, $p_{\nu_\mu} \equiv k_1$, $p_e \equiv k_2$, $p_{\bar\nu_e} \equiv k_3$, and the spins for $\{\mu, \nu_\mu, e, \bar\nu_e\}$ to be respectively s_0, s_1, s_2, s_3 (you will sum over spins, so their names are not important).

   (b) Compute the Feynman amplitude $i\mathcal{M}$ for this decay at tree level, in terms of spinors and gamma matrices.

   (c) Write out $|\mathcal{M}|^2$ in a form involving spinors and gamma matrices, without any "*" conjugation symbols. Take care to understand how spinor indices and Lorentz indices are contracted in this expression.

   (d) Take your expression for $|\mathcal{M}|^2$ and <u>average</u> over the initial muon spin, and <u>sum</u> over all the other particle spins. Use the relations

$$\sum_s u_s(p)\bar u_s(p) = (-\not p + m) \ , \qquad \sum_s v_s(p)\bar v_s(p) = (-\not p - m) \tag{5}$$

   to express $\langle|\mathcal{M}|^2\rangle$ (where $\langle\ \rangle$ denotes the spin averaging you performed) in terms of traces over gamma matrices. Note you will have a product of traces, with nontrivial Lorentz index contraction.

(e) Perform all the gamma matrix traces and express your result for $\langle|\mathcal{M}|^2\rangle$ as a polynomial in the variables of the problem, the four 4-momenta and the muon mass. You should find $\langle|\mathcal{M}|^2\rangle \propto (k_1 \cdot k_2)(k_3 \cdot p)$ before continuing to the next part of the problem; or if you cannot, assume that form before continuing.

(f) The formula for the differential decay width in the rest frame of the muon is

$$d\Gamma = \frac{d^3\mathbf{k}_1}{(2\pi)^3 2\omega_1}\frac{d^3\mathbf{k}_2}{(2\pi)^3 2\omega_2}\frac{d^3\mathbf{k}_3}{(2\pi)^3 2\omega_3}(2\pi)^4\delta(\omega_1+\omega_2+\omega_3-M)\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{\langle|\mathcal{M}|^2\rangle}{2M} \tag{6}$$

where the 4-momenta for the outgoing massless particles are $k_i^\mu = \{\omega_i, \mathbf{k}_i\}^\mu = \omega_i\{1, \hat{\mathbf{n}}_i\}^\mu$ where $\hat{\mathbf{n}}_i$ are unit 3-vectors. Compute the total width Γ by performing the integrations. Turn over page for hints.

Hints:

i. Express your formula for $\langle|\mathcal{M}|^2\rangle$ in terms of the ω_i and the $\hat{\mathbf{n}}_i$, and perform the integration over $d^3\mathbf{k}_3$.

ii. Write the remaining $d^3\mathbf{k}$ in spherical coordinates, with $|\mathbf{k}| = \omega$. You can then do three of the four remaining angular integrals for free, leaving integration over $d\omega_1$, $d\omega_2$ and $d\cos\theta_{12}$, where $\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \equiv \cos\theta_{12}$. ($\theta_{12}$ is the polar angle between the electron and the muon neutrino).

iii. You still have a factor $\delta(\omega_1+\omega_2+\omega_3-M)$, but because of 3-momentum conservation, ω_3 should be expressed as a function of ω_1, ω_2 and $\cos\theta_{12}$. After you have done so, do the $d\cos\theta_{12}$ integration, eliminating this remaining δ function.

iv. You are left with the ω_1 and ω_2 integrals. The only tricky part left is to figure out the bounds on these integrals. What is the range of values ω_1 can take? It can be as small as zero, but not as large as M. Once you have figured out the upper bound $\omega_1^{\max}$, then ask: given a value for ω_1, what is the allowed range for ω_2? Once you figure this out, the remaining integrals

$$\int_0^{\omega_1^{\max}}\int_{\omega_2^{\min}}^{\omega_2^{\max}} d\omega_1 d\omega_2$$

are readily computed, where in principle, $\omega_2^{\min}$ and $\omega_2^{\max}$ are functions of ω_1.

(g) Using the values for the muon mass M and G_F, compute Γ in MeV. The compute the muon lifetime $\tau = 1/\Gamma$ in seconds, using the conversion $(1\,\mathrm{GeV})^{-1} = 6.58 \times 10^{-25}$ sec. How does your answer compare with the experimental result, $\tau = 2.20 \times 10^{-6}$ sec?