Problem set #2 (1b revised 1/18/14), Physics 571 **Due Tuesday 1/21/2014**

---

1. With $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$, compute how $\bar{\Psi}\sigma^{\mu\nu}\Psi$ transforms under C, P and T.

2. On problem set #1 we saw that all fermion bilinears can be expressed as a linear combination of the sixteen bilinears $\bar{\Psi}\Gamma_a\Psi$ where Γ_a is one of the matrices $\{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$ and $\mathrm{Tr}\,\Gamma_a^\dagger\Gamma_b = 4\delta_{ab}$.

   (a) Express $\bar{\Psi}\sigma_{\mu\nu}\gamma_5\Psi$ in terms of the $\bar{\Psi}\Gamma_a\Psi$. (Hint: the indices really do tell you how the object transforms under the Lorentz group, so you cannot expect $\bar{\Psi}\sigma^{\mu\nu}\gamma_5\Psi$ to contain a term like $\bar{\Psi}\Psi$.)

   (b) *Revised*. Define the operator $S_3 = \int d^3x \bar{\Psi}\sigma^{12}\Psi$. What is its action on electron states $|\mathbf{p}, s, \mp\rangle$, where $\mp$ refers to electric charge (eg, "$-$"=particle, "$+$"= antiparticle), $s = \pm$ is the spin index, and $\mathbf{p}$ is the 3-momentum assumed to be nonrelativistic: $|\mathbf{p}| \ll m$. You can work out S_3 in the chiral basis, using the u and v spinors given in problem set #1, but using indices $s = 1, -1$ instead of $s = 1, 2$ so that you can express the s dependence of your answer simply.

3. Consider a theory of a massless Dirac fermion, $\mathcal{L} = \bar{\Psi}i\,\partial\!\!\!/\Psi$. We know that this is equivivalent to a theory of two uncoupled massless Weyl fermions, and therefore should have an independent phase rotation symmetry for each one. In the Dirac language, the two phase symmetries are $\Psi \to e^{i\alpha}\Psi$ and $\Psi \to e^{i\beta\gamma_5}\Psi$. This latter symmetry is called a chiral symmetry.

   (a) Show that $\Psi \to e^{i\beta\gamma_5}\Psi$ is a symmetry of the massless Dirac theory, in a basis independent manner.

   (b) What is the Noether current j_5^μ corresponding to this symmetry?

   (c) Express the conserved charge $Q_5 = \int d^3x j_5^0$ in terms of the b and d creation and annihilation operators. Can you interpret what this chiral charge is physically?

4. Consider the following Lagrangian for two real scalar fields $\phi_{1,2}$ interacting with a Dirac fermion,

$$\begin{aligned} \mathcal{L} &= \bar{\Psi}(i\,\partial\!\!\!/ - m)\Psi + \frac{1}{2}\left(-\partial_\mu\phi_1\partial^\mu\phi_1 - \partial_\mu\phi_2\partial^\mu\phi_2 - m_1^2\phi_1^2 - m_2^2\phi_2^2\right) \\ &\quad + g\phi_1\bar{\Psi}\Psi + ih\phi_2\bar{\Psi}\gamma_5\Psi \ . \end{aligned} \tag{1}$$

   (a) Show that the interaction terms are Hermitian if coupling constants g and h are real.

   (b) Show that this Lagrangian is invariant under C, P and T, and make a table for how ϕ_1 and ϕ_2 must transform under these three discrete symmetries.

   (c) Suppose if instead of just the scalar mass terms, we add to $\mathcal{L}$ a more general scalar interaction $-V(\phi_1, \phi_2)$, where V is a local hermitian polynomial consisting only of terms quartic in scalar fields (no derivatives). Which of the following symmetries can you break with the proper choice of V: C, P, T, CP, CT, PT, CPT?

   (d) Consider the special case of the above Lagrangian with $m = 0$, $m_1 = m_2 = \mu$, $h = g$. For these values of the couplings the theory has an exact chiral $U(1)$ symmetry where $\Psi \to e^{i\alpha\gamma_5}\Psi$ and the scalar fields also transform. How must the scalars transform for this to be a symmetry? If you add $V(\phi_1, \phi_2)$ to this theory (consisting only of quartic terms, as above), what is the most general form of V that respects this chiral $U(1)$ symmetry?

5. Prove eq. 42.10 in Srednicki:

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)} f(p)}{p^2 + m^2 - i\epsilon} = i\theta(x^0 - y^0) \int \widetilde{dp} e^{ip(x-y)} f(p) + i\theta(y^0 - x^0) \int \widetilde{dp} e^{-ip(x-y)} f(-p) \tag{2}$$

where in the $\int \widetilde{dp}$ integrals, $p^0 = \omega = \sqrt{|\mathbf{p}|^2 + m^2}$, and

$$\theta(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases} . \tag{3}$$

Be clear about how you are evaluating contour integrals.