Problem set #1, Physics 571 **Due Tuesday 1/14/2013**

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1. Compute:

   (a) $\mathrm{Tr}\,(\not a)$
   (b) $\mathrm{Tr}\,(\gamma_5)$
   (c) $\mathrm{Tr}\,(\gamma_5 \not a)$
   (d) $\mathrm{Tr}\,(\not a \not b)$
   (e) $\mathrm{Tr}\,(\gamma_5 \not a \not b)$
   (f) $\mathrm{Tr}\,(\not a \not b \not c)$
   (g) $\mathrm{Tr}\,(\gamma_5 \not a \not b \not c)$
   (h) $\mathrm{Tr}\,(\not a \not b \not c \not d)$
   (i) $\mathrm{Tr}\,(\gamma_5 \not a \not b \not c \not d)$

   where $\not a \equiv a_\mu \gamma^\mu$. Do not use a particular basis for the gamma matrices, but just the anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$, $\{\gamma_5, \gamma^\mu\} = 0$, $\gamma_5^2 = 1$.

   Hints: For some of these objects it is useful to use the cyclic property of a trace; for others it can help to insert $(\gamma_5)^2 = 1$ and make use of $\{\gamma_5, \gamma^\mu\} = 0$. The trace of four γ's and a γ_5 will involve the totally antisymmetric ϵ tensor: $\epsilon^{0123} = 1 = -\epsilon_{0123}$. Note that the 4-index ϵ tensor (in contrast to the 3-index version) is *anti*-cyclic, $\epsilon^{bcda} = -\epsilon^{abcd}$.

2. Define the sixteen matrices $\Gamma_a = \{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$ with $a = 1 \ldots, 16$, accounting for all the different values of the μ and ν indices. Show that $\mathrm{Tr}\,\Gamma_a^\dagger \Gamma_b = 4\delta_{ab}$. You can assume γ^0 is Hermitian, while γ^i is anti-Hermitian, which is consistent with $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. Explain therefore why a complete set of bilinears on can make from $\bar\Psi$ and Ψ are the sixteen quantities $\bar\Psi \Gamma_a \Psi$. How do these tensors transform under the Lorentz group? (eg as scalar, four vector...).

3. Consider the spinors **in the chiral basis** in which γ_5 is diagonal:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} , \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{1}$$

   where $\sigma^\mu = \{1, \vec\sigma\}$ and $\bar\sigma^\mu = \sigma_2(\sigma^\mu)^T\sigma_2 = \{1, -\vec\sigma\}$, and

$$u_1(\mathbf{p}) = \frac{-\not p + m}{\sqrt{2(m+\omega)}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \quad u_2(\mathbf{p}) = \frac{-\not p + m}{\sqrt{2(m+\omega)}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

$$v_1(\mathbf{p}) = \frac{\not p + m}{\sqrt{2(m+\omega)}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} , \quad v_2(\mathbf{p}) = \frac{\not p + m}{\sqrt{2(m+\omega)}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{3}$$

   where $p^0 = \omega = \sqrt{m^2 + |\mathbf{p}|^2}$ with $\mathbf{p}$ being the 3-momentum, and $\not p = p_\mu \gamma^\mu$. Also define spinors $\bar u_r = u_r^\dagger \gamma^0$ and $\bar v_r = v_r^\dagger \gamma^0$.

(a) Show that

$$(\not{p}+m)u_r(\mathbf{p}) = (-\not{p}+m)v_r(\mathbf{p}) = 0 \tag{4}$$

(b) Define the spin operator $S_z = \sigma^{12}$, where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$; show that in the particle rest frame ($\mathbf{p} = 0$)

$$S_z u_1(0) = \frac{1}{2}u_1(0)\ , \quad S_z u_2(0) = -\frac{1}{2}u_2(0)\ , \quad S_z v_1(0) = -\frac{1}{2}v_1(0)\ , \quad S_z v_2(0) = \frac{1}{2}v_2(0)\ . \tag{5}$$

(c) Prove Srednicki eq. (38.21):

$$\bar{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{s's}\ , \qquad \bar{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) = 2p^\mu \delta_{s's} \tag{6}$$

(d) Prove Srednicki eq (38.23):

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = -\not{p}+m\ , \qquad \sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = -\not{p}-m \tag{7}$$