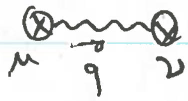
g-2 for the μ, e (anomalous magnetic moment)

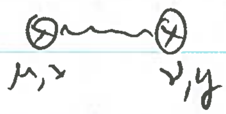
consider $\mathcal{L} = \mathcal{L}_{QED} + J_\mu A^\mu$

$J_\mu(x) \xleftrightarrow{FT} J_\mu(q)$ $\quad J_\mu(-q) = J_\mu^*(q)$ if $J_\mu(x)$ is real.



$$= i^2 J_\mu^*(q)\, J_\nu(q)\, \frac{-i}{q^2 - i\epsilon}\, g^{\mu\nu} \quad \text{(Feynman gauge)}$$

in position space; with time independent sources:

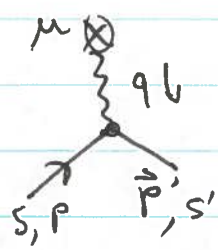


$$-i J_\mu(\vec{x})\, J_\nu(\vec{y}) \underbrace{\int \frac{d^3q}{(2\pi)^3} \cdot \frac{1}{\vec{q}^2 - i\epsilon}\, e^{i\vec{q}\cdot(\vec{x}-\vec{y})}}_{= \frac{1}{4\pi|\vec{x}-\vec{y}|}}$$

$$iT = \langle f|S-1|i\rangle \simeq \langle f|-iH_{int}|i\rangle \Rightarrow H_{int} = \frac{J_\mu(x)\, J_\nu(y)}{4\pi(\vec{x}-\vec{y})}$$

coulomb force.

Now consider scattering of a single source



$$= (+ie)(+iJ_\mu(q))\, \bar{u}_{s'}(\vec{p}')\, \gamma_\nu\, u_s(p) + \mathcal{O}(e^3)$$
$$\times \frac{-ig_{\mu\nu}}{q^2 - i\epsilon}$$

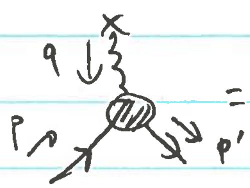
~~Note: classical eq of motion for A_μ: $\partial_\nu F^{\mu\nu} = J^\mu + \frac{1}{3}\partial^\mu \partial_\nu A^\nu$~~

Note: in Feynman gauge, classical eq

is $\partial_\nu F^{\mu\nu} - \partial_\nu \partial^\mu A^\nu = J^\mu$

$$\Rightarrow A^\mu_{cl}(x) = \frac{-1}{\Box} J^\mu \Rightarrow \frac{1}{q^2} J^\mu(q) = A^\mu_{cl}(q)$$

So $J_\mu(q) \cdot \frac{g^{\mu\nu}}{q^2 - i\epsilon} = A^\nu_{cl}$

So write 

$= i A^{cl}_\mu F^\mu$

$$F^\mu = e\, \bar{u}_{s'}(\vec{p}\,')\gamma^\mu u_s(\vec{p}) + O(e^3)$$

Gordon identity:

$$\bar{u}'\gamma_\mu u = \frac{-1}{2m}\bar{u}'(\not{p}'\gamma_\mu + \gamma_\mu \not{p})u$$ (since $\not{p}u = -mu$ etc)

$$\gamma_\mu \not{p} = \tfrac{1}{2}\{\gamma_\mu, \not{p}\} + \tfrac{1}{2}[\gamma_\mu, \not{p}]$$

$$= -p_\mu - i\sigma_{\mu\nu}p^\nu$$

so $$\boxed{e\,\bar{u}_{s'}(\vec{p}\,')\gamma^\mu u(p) = \frac{e}{2m}\bar{u}_{s'}(\vec{p}\,')\left[(p+p')^\mu - i\sigma^{\mu\nu}(p'-p)_\nu\right]u_s(\vec{p})}$$

Take non-relativistic limit $p \to (m, \vec{0})$, $p' \to (m, \vec{0})$
keeping terms just linear in $q = p' - p$

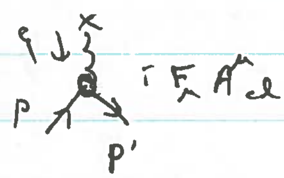
Dirac basis: $\gamma^0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

so $\sigma^{0i} = -i \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix}$ $\sigma^{ij} = \varepsilon_{ijk} \underbrace{\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}}_{\equiv \Sigma^k}$

$$u_s(\vec{p}) \xrightarrow[\vec{p} \to 0]{} \begin{pmatrix} \chi_s \\ \chi_s \end{pmatrix} \qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So to linear order in q:



$$(p+p')^\mu = 2m\delta^\mu_0$$
$$(p-p')^\mu = \vec{q}_i \delta^\mu_i \quad (\text{no sum on } i)$$

$$F^\mu \to \frac{e}{2m} \bar{u}(0) \Big[\delta^\mu_0 \cdot 2m + \sum_{j,k} \delta^\mu_i \, \varepsilon_{ijk} \Sigma^k q_j \Big] u(0)$$

$$i \varepsilon_{ijk} \Sigma^k q_j A^i_{cl}(q) \xrightarrow{FT} \vec{\Sigma} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\Sigma} \cdot \vec{B}$$

So $F^\mu = \bar{u}_{s'}(0) \left(e A^0_{cl} + \vec{\Sigma} \cdot \vec{B} \frac{e}{2m} \right) u_s(0)$

So Dirac eq (Tree level graphs) predicts

$$H = -\vec{\mu}\cdot\vec{B} \Rightarrow \vec{\mu} = \vec{\Sigma}\cdot\frac{e}{2m} \equiv \mu\vec{S}$$

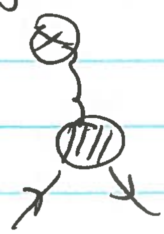
$\cancel{= \gamma\vec{\mu}}$ but $\vec{\Sigma} = 2\vec{S}$, so

$$\mu = \frac{e}{m} \quad \text{(Dirac)}$$

$$\equiv \frac{e}{2m}\cdot g \qquad g = 2 + O(e^3)$$

want to compute

Higher order calculation (Schwinger)



$: A^{cl}_{\mu} F^{\mu}$

general form of F^{μ}?

γ has $J^{PC} = 1^{--}$

Turn figure on side 

$\Rightarrow \gamma \to e^+e^-$ ... need e^+e^- in $J^{PC} = 1^{--}$

$\vec{J} = (\vec{L} + \vec{S})$ so need (i) $l=0, s=1$ (ii) $l=1, s=0$ (iii) $l=2\ s=1$
to have $J=1$ (iv) $l=1\ s=1$

under P: $P^{-1} b_s^\dagger(\vec{p}) P = i b_s^\dagger(-\vec{p})$

$P^{-1} d_s^\dagger(\vec{p}) P = i d_s^\dagger(-\vec{p})$

create e^+e^- pair with op

$$\Theta_{\ell,S} = \int \frac{d^3\vec{p}}{(2\pi)^3} \chi^S_{ss'} \phi_\ell(\vec{p}) \, b_s^\dagger(\vec{p}) d_{s'}^\dagger(-\vec{p})$$

$$\phi_\ell(-\vec{p}) = (-1)^\ell \phi_\ell(\vec{p})$$

$$\chi^S_{ss'} = (-1)^{S+1} \chi_{s's}$$

$$P^{-1} \Theta_{\ell,s} P = (-1)^{\ell+1}$$

$$C^{-1} b_s^\dagger(\vec{p}) C = d_s^\dagger(\vec{p})$$

$$C^{-1} d_s^\dagger(\vec{p}) C = b_s^\dagger(\vec{p})$$

So $C^{-1} \Theta_{\ell s} C = \Theta_{\ell s} \times \underbrace{(-1)}_{\substack{\text{fermi} \\ \{b^\dagger, d^\dagger\}=0}} \times \underbrace{(-1)^\ell}_{p \leftrightarrow -\vec{p}} \underbrace{(-1)^{1+S}}_{s \leftrightarrow s'}$

$$= (-1)^{\ell+S} \Theta_{\ell s}$$

ℓ	S	P	C	
0	1	-1	-1	✓
1	0	+1	-1	✗
1	1	+1	+1	✗
2	1	-1	-1	✓

∴ F^μ will depend on two functions of q^2:

So most general possible form for F^μ is

$$F^\mu = e\,\bar{u}_{s'}(p')\left[\gamma^\mu F_1(q^2) - i\frac{\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2)\right] u_s(p)$$

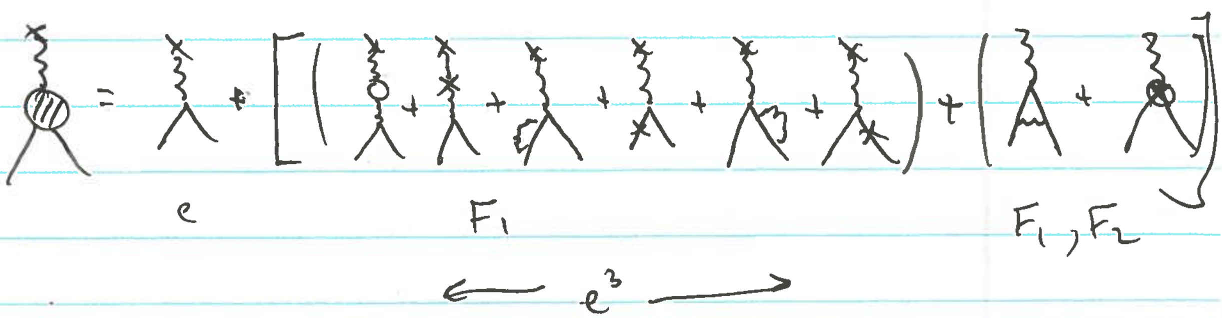
(we ruled out $\gamma^\mu\gamma_5$ and $\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}q_\nu = \sigma^{\mu\nu}\gamma_5 q_\nu$)
and q^μ, $q^\mu\gamma_5$

By charge conservation $F_1(0) = 1$

and ~~$F_2(0)$~~ $\vec{\mu} = \frac{e}{m}(1+F_2(0))$

$$\boxed{g-2 = 2F_2(0)}$$

Need to compute $F_2(0)$ to order (e^2)



can ignore all but

(c.t.  cannot affect F_2 as there is no $F_{\mu\nu}\bar{q}\sigma^{\mu\nu}q$ op. in $\mathcal{L}$)

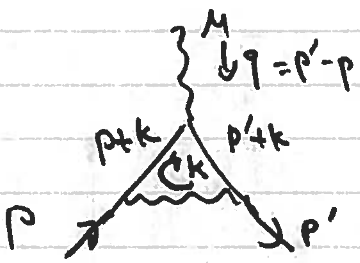
So to get $F_2(0)$ need only compute relevant part of

No i due to def of F_2

$$F^\mu = e\,(ie)^2\cdot(-i)^3\cdot$$

$$\times \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-i\epsilon}\cdot\frac{1}{(p+k)^2+m^2-i\epsilon}\,\frac{1}{(p'+k)^2+m^2-i\epsilon}$$

$$\times\,\bar{u}'\gamma_\lambda(-\not{p}'-\not{k}+m)\gamma^\mu(-\not{p}-\not{k}+m)\gamma_\lambda u$$



$$= \cdots i e^3 \int \frac{d^4k}{(2\pi)^4}\,\frac{N}{D}$$

$$N = \bar{u}'\gamma_\lambda(-\not{p}'-\not{k}+m)\gamma^\mu(-\not{p}-\not{k}+m)\gamma_\lambda u$$

Use: $(-\not{p}+m)\gamma_\lambda u = \gamma_\lambda(\not{p}+m)u - \{\gamma_\lambda,\not{p}\}u$

$$= 0 + 2p_\lambda u$$

$$\bar{u}'\gamma_\lambda(-\not{p}'+m) = 2\bar{u}\,p'_\lambda$$

So $N = \bar{u}'(2p'_\lambda - \gamma_\lambda\not{k})\gamma^\mu(2p^\lambda - \not{k}\gamma^\lambda)u$

We are looking for $F_2(q^2)\sigma^{\mu\nu}q_\nu$ term $= O(q) \to 0$ as $p \to p'$.

So drop $p'\cdot p = \frac{-q^2}{2} - m^2$ (contains no $O(q)$)

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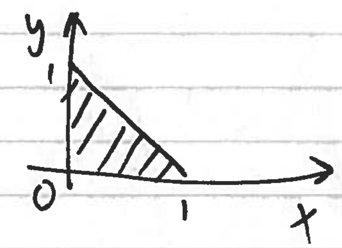
Stuff we want

$$N \simeq \bar{u}'\left(-2\not{p}\not{k}\gamma^\mu - 2\gamma^\mu\not{k}\not{p}' + \gamma_\lambda\not{k}\gamma^\mu\not{k}\gamma^\lambda\right)u$$

use identity: $\gamma_\lambda \not{k}\gamma^\mu\not{k}\gamma^\lambda = 2\not{k}\gamma^\mu\not{k}$

so $$\boxed{N \simeq \bar{u}'\left(-2\not{p}\not{k}\gamma^\mu - 2\gamma^\mu\not{k}\not{p}' + 2\not{k}\gamma^\mu\not{k}\right)u}$$

denominator: $$\frac{1}{abc} = 2\int_\Delta dx\,dy\,\frac{1}{[ax+by+c(1-x-y)]^3}$$

$\Delta =$ 

~~so~~ $\cancel{\frac{1}{D} = 2\int_\Delta dx\,dy}$

$$\frac{1}{D} = \frac{1}{k^2 - i\epsilon}\;\frac{1}{(p+k)^2+m^2-i\epsilon}\;\frac{1}{(p'+k)^2+m^2-i\epsilon}$$

$$= \underset{c}{\frac{1}{k^2-i\epsilon}}\;\underset{a}{\frac{1}{2p\cdot k + k^2 - i\epsilon}}\;\underset{b}{\frac{1}{2p'\cdot k + k^2 - i\epsilon}}$$

$p^2 = p'^2 = -m^2$

$$= 2\int_\Delta dx\,dy\,\frac{1}{[k^2 + 2p\cdot k\,x + 2p'\cdot k\,y - i\epsilon]^3}$$

shift: $k' = k + px + p'y$

$$\Rightarrow \frac{1}{D} = 2\int_\Delta \frac{1}{[k'^2 - (px+p'y)^2]^3}$$

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$$= 2\int_\Delta dx\,dy\ \left[k'^2 + m^2x^2 + m^2y^2 - 2p\cdot p'\,xy\right]^{-3}$$

again: $q^2 = (p-p')^2 = -2m^2 - 2p\cdot p'$

so since we want only to $\mathcal{O}(q)$ part,

can replace $-2p\cdot p' = q^2 + 2m^2 \approx 2m^2$

left with $$\boxed{\frac{1}{D} \approx 2\int_\Delta dx\,dy\ \left[k'^2 + m^2(x+y)^2\right]^{-3} + \mathcal{O}(q^2)}$$

Back to N: have to shift $k = k' - px - p'y$. Since D ~~N = ...~~ is even in k', can drop terms in N which are odd in k':

$$N \approx \bar{u}'\Big[+2\not{p}(\not{p}x+\not{p}'y)\gamma^\mu + 2\gamma^\mu(\not{p}x+\not{p}'y)\not{p}'$$
$$+ 2\not{k}'\gamma^\mu\not{k}' + 2(\not{p}x+\not{p}'y)\gamma^\mu(\not{p}x+\not{p}'y)\Big]u$$

Note: $\int d^4k'\, \underset{\wedge}{f(k'^2)}\, k'_\mu k'_\nu = \frac{1}{4}\int d^4k'\, f(k'^2)\, k'^2\, g_{\mu\nu}$ any function $f(k'^2)$

so $\not{k}'\gamma^\mu\not{k}' \to \frac{1}{4}k'^2\gamma_\alpha\gamma^\mu\gamma^\alpha = \frac{1}{2}k'^2\gamma^\mu$
↳ only contributes to F_1

so ~~N = ...~~ Also: $\not{p}\not{p} = -p^2 = m^2$ ... so $\not{p}\not{p}\gamma^\mu \propto \gamma^\mu$
↳ only contributes to F_1

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So the parts we are interested in for F_2 are

$$N \simeq 2\,\bar{u}'\left[y\,\not{p}\not{p}'\gamma^\mu + x\,\gamma^\mu\not{p}\not{p}' + (\not{p}x+\not{p}'y)\gamma^\mu(\not{p}x+\not{p}'y)\right]u$$

Use $\bar{u}'\not{p}\not{p}' = \bar{u}'\left(\{\not{p},\not{p}'\} - \not{p}'\not{p}\right)\gamma^\mu$

$= \bar{u}'\left(\underbrace{-2p\cdot p'}_{} + m\not{p}\right)\gamma^\mu$

contributes to F_1 ... drop

similarly $\gamma^\mu\not{p}\not{p}'u \to m\gamma^\mu\not{p}'u$

$$N \simeq 2\bar{u}'\left[my\,\not{p}\gamma^\mu + mx\,\gamma^\mu\not{p}' + (\not{p}x+\not{p}'y)\gamma^\mu(\not{p}x+\not{p}'y)\right]u$$

Next use $p' = p+q$, $p = p'-q$ $\quad \not{p}u = -mu$, $\bar{u}'\not{p}' = -m\bar{u}'$

... and drop F_1 terms

$$N \simeq 2\bar{u}'\left[-my\,\not{q}\gamma^\mu + mx\,\gamma^\mu\not{q} + (-m(x+y) - \not{q}x)\gamma^\mu(-m(x+y) + \not{q}y)\right]u$$

$$\simeq 2\bar{u}'\left[-my\,\not{q}\gamma^\mu + mx\,\gamma^\mu\not{q} + x(x+y)\cdot m\not{q}\gamma^\mu - y(x+y)m\gamma^\mu\not{q}\right]u$$

... dropping $O(q^2)$ term.

Since D is symmetric in x,y we can symmetrize N in x,y:

$$x \to \tfrac{1}{2}(x+y)$$
$$y \to \tfrac{1}{2}(x+y)$$

$$N \simeq \bar{u}'\, m(x+y)\,[\gamma^\mu, \not{q}] - (x+y)^2 m [\gamma^\mu, \not{q}]\big) u$$

$$= m(x+y)(1-(x+y))\ \bar{u}' \underbrace{[\gamma^\mu, \not{q}]}_{= -2i q_\nu \sigma^{\mu\nu}} u$$

Recall: $F^\mu = e\bar{u}\left[\gamma^\mu F_1 - \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2\right] u = -ie^3 \int \frac{d^4k}{(2\pi)^4} \frac{N}{D}$

So $F_2(0) = \frac{2m}{e}(-ie^3)\left[2\int_\Delta dx\,dy \cdot 2m(x+y)(1-x-y)\cdot \int \frac{d^4k}{(2\pi)^4} \cdot \underbrace{\frac{1}{[k^2+m^2(x+y)^2]^3}}_{1/D}\right]$

$d^4k \to i\,d^4k_E$

$$= \frac{8m^2e^2}{(2\pi)^4} \cdot \underset{\substack{\uparrow \\ \text{solid angle} \\ \text{in 4D}}}{2\pi^2} \cdot \int_\Delta dx\,dy \overset{(x+y)(1-x-y)}{\vee} \underbrace{\int_0^\infty k^3 dk\, [k^2+m^2(x+y)^2]^{-3}}_{\frac{1}{4m^2(x+y)^2}}$$

$$= \frac{e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy\ \frac{1-x-y}{x+y} = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi} \qquad \alpha \equiv \frac{e^2}{4\pi}$$

So $\boxed{F_2(0) = \frac{\alpha}{2\pi}} \Rightarrow \mu = \frac{e}{m}\left(1 + \frac{\alpha}{2\pi}\right)$ or: $(g-2) = \frac{\alpha}{\pi}$

$+\mathcal{O}(\alpha^2)$.

This result:

$$a \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} = 0.0011614$$

$a^e_{exp} =$ 0.00115965218073(28) electron

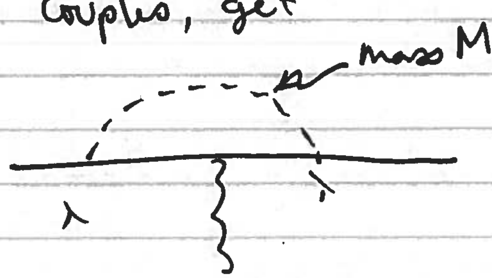
$a^\mu_{exp} =$ 0.0011659208 9(54)(33)

might be out of date now

discrepancy = few × ~~[illegible]~~ 10^{-6}

Note: $\left(\frac{\alpha}{2\pi}\right)^2 \sim 10^{-6}$.

~~[illegible]~~ μ interesting because if there is a heavy exotic particle that couples, get



expect $g-2 \sim \frac{\lambda^2}{8\pi^2} \cdot \frac{m^2}{M^2}$

bigger effect for bigger m (μ vs e)

~~at~~ $\delta a^\mu_{exp} \sim 10^{-10} \sim \frac{\alpha}{2\pi} \cdot 10^{-7}$, if $\frac{\lambda^2}{8\pi^2} \sim \frac{\alpha}{2\pi}$)

then experiment is sensitive to $\frac{m^2}{M^2} \sim 10^{-7}$

or $M \sim 10^{3.5} m_\mu \sim 340$ GeV.

Note:  loop must be dominated by $k \sim M$

So in , heavy particle prop gives $\sim \frac{m^2}{M^2}$ suppression.