Physics 570 — Handout on Dimensional Regularization

Consider the following integral in n dimensions with a *Euclidian* metric:

$$I_1 \equiv \int \mathrm{d}^n k \; \frac{1}{(k^2+a^2)^r} \; .$$

We may evaluate this making in terms of the Γ function:

$$\alpha^{-s}\Gamma(s) = \int_0^\infty \mathrm{d}x \; x^{s-1} \; \mathrm{e}^{-\alpha x}.$$

Then

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(r)} \int \mathrm{d}^n k \; \int_0^\infty \mathrm{d}x \; x^{r-1} \; \mathrm{e}^{-x(k^2+a^2)} \\ &= \frac{\pi^{n/2}}{\Gamma(r)} \int_0^\infty \mathrm{d}x \; x^{r-1-n/2} \; \mathrm{e}^{-xa^2} \\ &= \pi^{n/2} a^{n-2r} \frac{\Gamma(r-n/2)}{\Gamma(r)} \end{aligned}$$

Another useful integral is

$$I_2 \equiv \int \mathrm{d}^n k \; \frac{k^2}{(k^2+a^2)^r} \; .$$

To get this we define

$$\begin{aligned} I_1(\alpha) &\equiv \int \mathrm{d}^n k \; \frac{1}{(\alpha k^2+a^2)^r} \; ; \\ &= \alpha^{-n/2} I_1 \end{aligned}$$

then by differentiating by α and setting $\alpha = 1$ we find

$$I_2 = \frac{n\pi^{n/2} a^{n-2r+2}}{2(r-1)} \frac{\Gamma(r-1-n/2)}{\Gamma(r-1)} \; .$$

Finally note that

$$\begin{aligned} I_3^{\mu\nu} &\equiv \int \mathrm{d}^n k \; \frac{k^\mu k^\nu}{(k^2+a^2)^r} \\ &= \frac{\delta^{\mu\nu}}{n} I_2 \end{aligned} \; .$$

Useful Properties of Γ Functions

Gamma functions have the property $\Gamma(z+1) = z\Gamma(z)$, with $\Gamma(1) = 1$. Thus for integers $n \geq 1$,

$$\Gamma(n+1) = n!, \qquad n \geq 1 \ .$$

Also useful is the value

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \ .$$

The Gamma function is singular for non-positive integer arguments. Near these singularities it can be expanded as

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] \ ,$$

where

$$\psi(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \ ,$$
$$\gamma = 0.5772\ldots$$

In particular,

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + \gamma - 1$$
$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma$$

Useful consequences:

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{1}{q^2+m^2} = \frac{m^2}{16\pi^2}\left[-\frac{1}{\epsilon} + \gamma - 1 - \ln 4\pi + \ln(m^2/\mu^2)\right]$$

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{1}{(q^2+m^2)^2} = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln(m^2/\mu^2)\right]$$

$$\mu^{2\epsilon}\int \frac{\mathrm{d}^{4-2\epsilon}q}{(2\pi)^{4-2\epsilon}}\frac{q^2}{(q^2+m^2)^2} = \frac{m^2}{8\pi^2}\left[-\frac{1}{\epsilon} + \gamma - \frac{1}{2} - \ln 4\pi + \ln(m^2/\mu^2)\right]$$