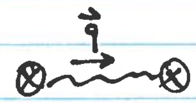
Magnetic moment of the electron

$\mathcal{L} = \mathcal{L}_{QED} + J_\mu A^\mu$ work in Feynman gauge,

$$\mu \sim\sim \nu = \frac{-i}{k^2 - i\epsilon} g_{\mu\nu}$$

$$i\mathcal{M} = \langle f | S - 1 | i \rangle = \langle f | -i H_{int} | i \rangle$$

Look at  $= (i)^2 J^*_\mu(\vec{q})\, J_\nu(\vec{q}) \frac{-i}{+\vec{q}^2 - i\epsilon} g^{\mu\nu}$

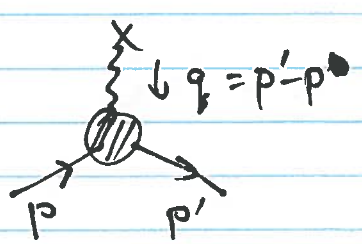
$$= J_\mu(\vec{x})\, J^\mu(\vec{y}) \int \frac{d^3 q}{(2\pi)^4} e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \frac{+i}{\vec{q}^2 - i\epsilon} =$$

Assume static sources, $J(q) \propto \delta(q^0)$

$$= \frac{i J_\mu(\vec{x})\, J^\mu(\vec{y})}{4\pi |\vec{x}-\vec{y}|} \Rightarrow H_{int} = -\frac{J_\mu(\vec{x}) J^\mu(\vec{y})}{4\pi |\vec{x}-\vec{y}|}$$

$J_\mu(\vec{x}) J^\mu(\vec{y}) \to -\rho(\vec{x})\rho(\vec{y}) \Rightarrow$ get repulsive coulomb.

Now consider an electron scattering off J



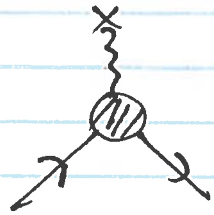
$$= (ie)(iJ_\mu)\left(\frac{-ig^{\mu\nu}}{q^2 - i\epsilon}\right) \bar{u}_{s'}(p')\gamma_\nu u_s(p) + \mathcal{O}(e^3)$$

$$= +ie \frac{J_\mu(q)}{q^2 - i\epsilon} \bar{u}_{s'}(p')\gamma^\mu u_s(p) + \mathcal{O}(e^3)$$

<u>Note</u> $J_\mu(q) \cdot \frac{1}{q^2 - i\epsilon} = A^{cl}_\mu$: $\partial_\nu \partial^\nu A^{cl}_\mu = -J_\mu$ (Maxwell)

$\Rightarrow A^{cl} = -J_\mu / \partial_\nu \partial^\nu = J_\mu / q^2$.

So write



$$\equiv \; +i\, A^{cl}_\mu F^\mu$$

$$F^\mu = e\, \bar{u}_{s'}(p')\gamma^\mu u_s(p) + \mathcal{O}(e^3)$$

Use Gordon identity

$$\bar{u}'\gamma_\mu u = \frac{-1}{2m}\bar{u}'\left(\not{p}'\gamma_\mu + \gamma_\mu \not{p}\right)u$$

$$\gamma_\mu \not{p} = \tfrac{1}{2}\{\gamma_\mu, \not{p}\} + \tfrac{1}{2}[\gamma_\mu, \not{p}]$$

$$= -p_\mu - i\sigma_{\mu\nu}p^\nu$$

put together, get Srednicki eq 38.18:

$$e\,\bar{u}(p')\gamma^\mu u(p) = \frac{e}{2m}\bar{u}'\left[(p'+p)^\mu - i\sigma^{\mu\nu}(p'-p)_\nu\right]u$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Take $p \to (m, \vec{0})$ keep $q = p' - p$ to
$p' \to (m, \vec{0})$ linear order.

$$\cancel{\frac{(p'+p)^\mu}{2m} \to \delta^\mu_0}$$

$$\cancel{-i\sigma^{\mu\nu}(p'-p)_\nu \to \delta^\mu_i \delta^\nu_j \varepsilon_{ijk} S_k q_\nu}$$

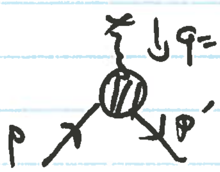
Note that $u_s(\vec{p}=0)=\sqrt{m}\begin{pmatrix}\chi_s\\ \chi_s\end{pmatrix}$ $\qquad \chi_+=\begin{pmatrix}1\\0\end{pmatrix},\ \chi_-=\begin{pmatrix}0\\1\end{pmatrix}$

$\gamma^0=\begin{pmatrix} & 1\\ 1 & \end{pmatrix} \qquad \gamma^i=\begin{pmatrix} & \sigma^i\\ -\sigma^i & \end{pmatrix}$

so $\sigma^{0i}=\frac{i}{2}[\gamma^0,\gamma^i]=-i\begin{pmatrix}\sigma^i & \\ & -\sigma^i\end{pmatrix}$ $\qquad \bar{u}_{s'}(0)\,\sigma^{0i}\,u_s(0)=0$

$$\sigma^{ij}=\frac{i}{2}[\gamma^i,\gamma^j]=-\frac{i}{2}\begin{pmatrix}[\sigma^i,\sigma^j] & \\ & [\sigma^i,\sigma^j]\end{pmatrix}=+\epsilon_{ijk}\underbrace{\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}}_{\equiv\Sigma^k}$$

so get



$= i A^{cl}_\mu(q)\, F^\mu$

$$F^\mu \underset{\text{tree}}{\Rightarrow} \frac{e}{2m}\,\bar{u}'(0)\left[\delta^\mu_0\cdot 2m + i\sum_{i,j}\delta^\mu_i\delta^\nu_j(\epsilon_{ijk}\Sigma^k)q_\nu\right.$$

$(p,p'\to m,\vec{0})$ $\qquad i q_j A_i\, \epsilon_{ijk}$

$= \text{F.T. of } +\vec\nabla\times\vec{A} = +\vec{B}$

so get

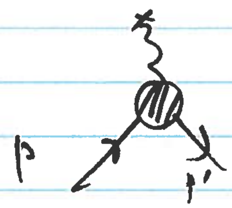
$= \bar{u}'(0)\left[eA^0+\vec\Sigma\cdot\vec{B}\cdot\frac{e}{2m}\right]u(0)$

compare 1st term = charge, 2nd = Pauli $H'=-\vec\mu\cdot\vec{B}$

so Dirac finds $\vec\mu=\frac{e}{2m}\vec\Sigma \equiv \mu\left(\frac{\vec\Sigma}{2}\right) \leftarrow$ spin operator.

So $\mu = \frac{e}{m} + \mathcal{O}(e^3)$ or: $\mu = g \cdot \frac{e}{2m}$, $g = 2 + \mathcal{O}(e^3)$.

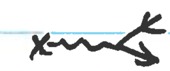
Higher order:



$= i A^{cl}_\mu F^\mu$

most general form of F^μ?

Turn diagram on side:



$\gamma \to e^+e^-$

$\gamma : J^{PC} = 1^{--}$

e^+e^- state: has spin $S=0$ or 1, so can only couple to γ in $l=0,1,$ or 2:

	l	S	P	C	
$J=1$:	0	1	-1	-1	
	1	0	$+1$	-1	← NOT ALLOWED FOR γ COUPLING
	2	1	-1	-1	

e^+e^- has intrinsic $P = -1 \times (-1)^l$

So there must be two independent "form factors":

$$F^\mu \equiv e\,\bar{u}'\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2)\right] u$$

$q^2 \to 0$: must have $F_1(0) \to 1$ says electron has charge e

(5)

so in nonrelativistic limit, we see we have

$$\vec{\mu} = \frac{e}{2m}\left[1+F_2(0)\right]\vec{\Sigma}$$

$$\equiv \mu\,\frac{\vec{\Sigma}}{2} \quad \Rightarrow \quad \boxed{\mu = \frac{e}{m}\left[1+F_2(0)\right]}$$

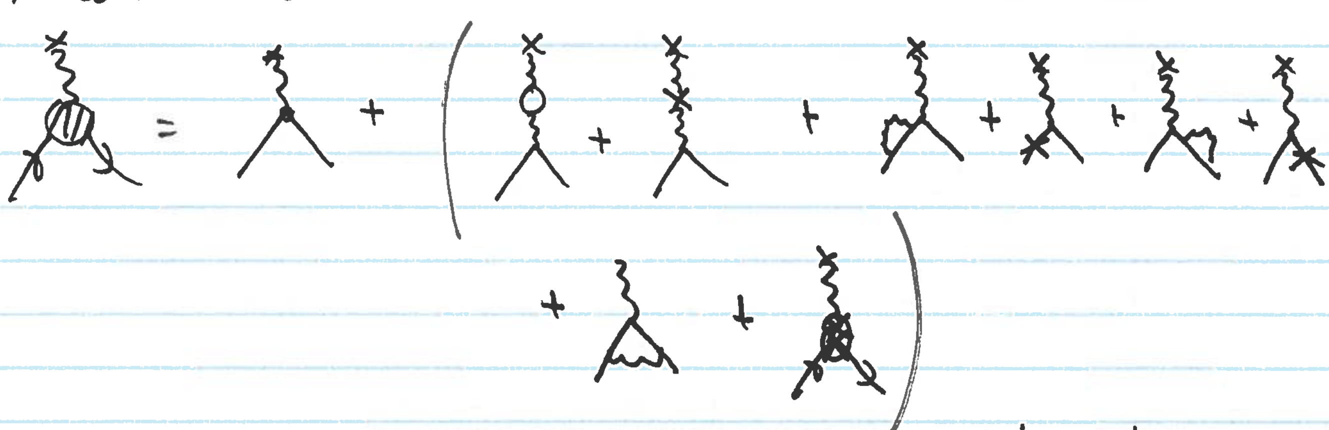
$$\cancel{= g\frac{e}{2m}\frac{\vec{\Sigma}}{2} \Rightarrow (g-2) = 2F_2(0)}$$

$$= g\frac{e}{2m}\frac{\vec{\Sigma}}{2} \quad \Rightarrow \quad \boxed{(g-2) = 2F_2(0)}$$

"anomalous magnetic moment"

so: we want to compute $F_2(0)$ to $\mathcal{O}(e^2)$. (Schwinger)

Need to look at



But Note: * 1st 6 $\mathcal{O}(e^3)$ graphs only contribute to $\bar{u}'\gamma^\mu u$

$\Rightarrow$ they contribute to $F_1(q^2)$

* The counter term  can only contribute to $\bar{\Psi}\gamma^\mu\Psi A_\mu$

or the F_1 term, since there is no $\bar{\Psi}\sigma^{\mu\nu}F_{\mu\nu}\Psi$ term in $\mathcal{L}$

6

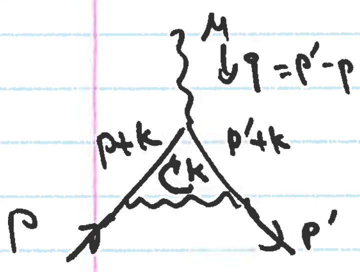
So to get $F_2(0)$ need only compute relevant part of

No i due to def of F_2

$$F^\mu = e\,(ie)^2\cdot(-i)^3\cdot$$

$$\times \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-i\epsilon}\,\frac{1}{(p+k)^2+m^2-i\epsilon}\,\frac{1}{(p'+k)^2+m^2-i\epsilon}$$

$$\times\, \bar u'\gamma_\lambda(-\not p'-\not k+m)\gamma^\mu(-\not p-\not k+m)\gamma_\lambda u$$



$$= \dots i e^3 \int \frac{d^4k}{(2\pi)^4}\,\frac{N}{D}$$

$$N = \bar u'\gamma_\lambda(-\not p'-\not k+m)\gamma^\mu(-\not p-\not k+m)\gamma_\lambda u$$

Use: $(-\not p+m)\gamma_\lambda u = \gamma_\lambda(\not p+m)u - \{\gamma_\lambda,\not p\}u$

$$= 0 + 2p_\lambda u$$

$$\bar u'\gamma_\lambda(-\not p'+m) = 2\bar u p'_\lambda$$

So $N = \bar u'(2p'_\lambda - \gamma_\lambda \not k)\gamma^\mu(2p^\lambda - \not k\gamma^\lambda)u$

We are looking for $F_2(q^2)\sigma^{\mu\nu}q_\nu$ term $= O(q) \to 0$ as $p \to p'$.

So drop $p'\cdot p = \frac{-q^2}{2} - m^2$ (contains no $O(q)$)

F_1

Stuff we want

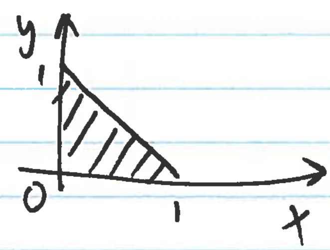
$$N \overset{\downarrow}{\simeq} \bar{u}'\left(-2\not{p}\not{k}\gamma^\mu - 2\gamma^\mu\not{k}\not{p}' + \gamma_\lambda \not{k}\gamma^\mu\not{k}\gamma^\lambda\right)u$$

use identity: $\gamma_\lambda \not{k}\gamma^\mu\not{k}\gamma^\lambda = 2\not{k}\gamma^\mu\not{k}$

so $$\boxed{N \simeq \bar{u}'\left(-2\not{p}\not{k}\gamma^\mu - 2\gamma^\mu\not{k}\not{p}' + 2\not{k}\gamma^\mu\not{k}\right)u}$$

denominator: $$\frac{1}{abc} = 2\int_\Delta dx\,dy \frac{1}{[ax+by+c(1-x-y)]^3}$$

$\Delta =$



~~so~~ $\cancel{\frac{1}{D} = 2\int_\Delta dx\,dy}$

$$\frac{1}{D} = \frac{1}{k^2-i\epsilon}\,\frac{1}{(p+k)^2+m^2-i\epsilon}\,\frac{1}{(p'+k)^2+m^2-i\epsilon}$$

$$= \underbrace{\frac{1}{k^2-i\epsilon}}_{c}\,\underbrace{\frac{1}{2p\cdot k+k^2-i\epsilon}}_{a}\,\underbrace{\frac{1}{2p'\cdot k+k^2-i\epsilon}}_{b}$$

$p^2 = p'^2 = -m^2$

$$= 2\int_\Delta dx\,dy \frac{1}{[k^2 + 2p\cdot k\,x + 2p'\cdot k\,y - i\epsilon]^3}$$

shift: $k' = k + px + p'y$

$$\Rightarrow \frac{1}{D} = 2\int_\Delta \frac{1}{[k'^2 - (px+p'y)^2]^3}$$

$$= 2\int_\Delta dx\,dy\,\left[k'^2 + m^2x^2 + m^2y^2 - 2p\cdot p'\,xy\right]^{-3}$$

again: $q^2 = (p-p')^2 = -2m^2 - 2p\cdot p'$

so since we want only to $\mathcal{O}(q)$ part,

can replace $-2p\cdot p' = q^2 + 2m^2 \approx 2m^2$

left with $$\boxed{\frac{1}{D} \approx 2\int_\Delta dx\,dy\,\left[k'^2 + m^2(x+y)^2\right]^{-3} + \mathcal{O}(q^2)}$$

Back to N: have to shift $k = k' - px - p'y$. Since D is even in k', can drop terms in N which are odd in k':

$$N \approx \bar{u}'\Big[+2\not{p}(\not{p}x+\not{p}'y)\gamma^\mu + 2\gamma^\mu(\not{p}x+\not{p}'y)\not{p}' + 2\not{k}'\gamma^\mu\not{k}' + 2(\not{p}x+\not{p}'y)\gamma^\mu(\not{p}x+\not{p}'y)\Big]u$$

Note: $\int d^4k'\, f(k'^2)\, k'_\mu k'_\nu = \frac{1}{4}\int d^4k'\, f(k'^2)\, k'^2 g_{\mu\nu}$ any function $f(k'^2)$

so $\not{k}'\gamma^\mu\not{k}' \to \frac{1}{4}k'^2\gamma_\alpha\gamma^\mu\gamma^\alpha = \frac{1}{2}k'^2\gamma^\mu$

only contributes to F_1

so N Also: $\not{p}\not{p} = -p^2 = m^2 \ldots$ so $\not{p}\not{p}\gamma^\mu \propto \gamma^\mu$

only contributes to F_1

So the parts we are interested in for F_2 are

$$N \simeq 2\bar{u}'\left[y \not{p}\not{p}'\gamma^\mu + x\gamma^\mu \not{p}\not{p}' + (\not{p}x + \not{p}'y)\gamma^\mu(\not{p}x+\not{p}'y)\right] u$$

Use $\bar{u}'\not{p}\not{p}' = \bar{u}'\left(\{\not{p},\not{p}'\} - \not{p}'\not{p}\right)\gamma^\mu$

$$= \bar{u}'\left(\underbrace{-2p\cdot p'}_{} + m\not{p}\right)\gamma^\mu$$

contributes to F_1 ... drop

similarly $\gamma^\mu \not{p}\not{p}' u \to m\gamma^\mu \not{p}' u$

$$N \simeq 2\bar{u}'\left[my\not{p}\gamma^\mu + mx\gamma^\mu\not{p}' + (\not{p}x+\not{p}'y)\gamma^\mu(\not{p}x+\not{p}'y)\right] u$$

Next use $p' = p+q$, $p = p'-q$ $\quad \not{p}u = -mu$, $\bar{u}'\not{p}' = -m\bar{u}'$

... and drop F_1 terms

$$N \simeq 2\bar{u}'\left[-my\not{q}\gamma^\mu + mx\gamma^\mu\not{q} + (-m(x+y) - \not{q}x)\gamma^\mu(-m(x+y) + \not{q}y)\right] u$$

$$\simeq 2\bar{u}'\left[-my\not{q}\gamma^\mu + mx\gamma^\mu\not{q} + x(x+y)\cdot m\not{q}\gamma^\mu - y(x+y)m\gamma^\mu\not{q}\right] u$$

... dropping $O(q^2)$ term.

Since D is symmetric in x, y we can symmetrize N in x, y:

$$x \to \tfrac{1}{2}(x+y)$$
$$y \to \tfrac{1}{2}(x+y)$$

$$N \simeq \bar{u}'\, m(x+y)\,[\gamma^\mu, \not{q}] - (x+y)^2 m\,[\gamma^\mu, \not{q}]\big)\, u$$

$$= m(x+y)(1-(x+y))\;\bar{u}'\underbrace{[\gamma^\mu, \not{q}]}_{=-2i q_\nu \sigma^{\mu\nu}} u$$

Recall: $F^\mu = e\bar{u}\left[\gamma^\mu F_1 - \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2\right] u = -ie^3 \int \frac{d^4k}{(2\pi)^4} \frac{N}{D}$

So $F_2(0) = \frac{2m}{e}(-ie^3)\left[2\int_\Delta dx\,dy \cdot 2m(x+y)(1-x-y) \cdot \int \frac{d^4k}{(2\pi)^4} \cdot \underbrace{\frac{1}{[k^2+m^2(x+y)^2]^3}}_{1/D}\right.$

$d^4k \to i\, d^4k_E$

$$= \frac{8m^2e^2}{(2\pi)^4} \cdot \underset{\substack{\uparrow \\ \text{solid angle} \\ \text{in 4D}}}{2\pi^2} \cdot \int_\Delta dx\,dy\, \overset{(x+y)(1-x-y)}{\cdot} \underbrace{\int_0^\infty k^3 dk\,[k^2+m^2(x+y)^2]^{-3}}_{\frac{1}{4m^2(x+y)^2}}$$

$$= \frac{e^2}{4\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \frac{1-x-y}{x+y} = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi} \qquad \alpha \equiv \frac{e^2}{4\pi}$$

So $\boxed{\boxed{F_2(0) = \frac{\alpha}{2\pi}} \Rightarrow \mu = \frac{e}{m}\left(1 + \frac{\alpha}{2\pi}\right) \quad \text{or: } (g-2) = \frac{\alpha}{\pi}}$

$+\mathcal{O}(\alpha^2)$.

This result:

$$a \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} = 0.0011614$$

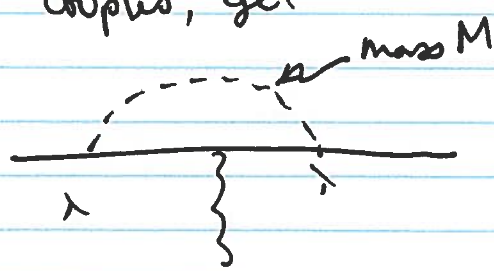
$$a^{e}_{exp} = 0.00115965218073(28)$$ electron

$$a^{\mu}_{exp} = 0.0011659208\,9(54)(33)$$

discrepancy = few × 10^{-6}

Note: $\left(\frac{\alpha}{2\pi}\right)^2 \sim 10^{-6}$.

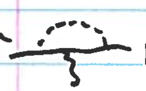
μ interesting because if there is a heavy exotic particle that couples, get



expect $g-2 \sim \frac{\lambda^2}{8\pi^2} \cdot \frac{m^2}{M^2}$

bigger effect for bigger m (μ vs e)

Note:  loop must be dominated by $k \sim M$

So in , heavy particle prop gives $\sim \frac{m^2}{M^2}$ suppression.

at $\delta a^{\mu}_{exp} \sim 10^{-10} \sim \frac{\alpha}{2\pi} \cdot 10^{-7}$, if $\frac{\lambda^2}{8\pi^2} \sim \frac{\alpha}{2\pi}$)

then experiment is sensitive to $\frac{m^2}{M^2} \sim 10^{-7}$

or $M \sim 10^{3.5} m_\mu \sim 340$ GeV.