Quantizing Spin 1 bosons. (including photons.)

~~Other~~ Spin 1: look at $(\frac{1}{2},\frac{1}{2}) = 4$ vector $= j=0 \oplus j=1$.

$A_\mu(x)$

General $\mathcal{L}$ quadratic in A:

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu + \alpha\, \partial_\mu A^\mu \partial_\nu A^\nu + \beta A_\mu A^\mu\right]$$

eqs of motion: $\partial^\mu \frac{\delta \mathcal{L}}{\delta \partial^\mu A^\nu} - \frac{\delta \mathcal{L}}{\delta A^\nu} = 0$.

$$-\partial^\mu(\partial_\mu A_\nu) - \alpha\, \partial_\nu(\partial^\mu A_\mu) + \beta A_\nu = 0$$

plane wave solution: $A_\mu = \varepsilon_\mu e^{ik\cdot x}$

constant vector "polarization vector"

$$+k^2 \epsilon_\nu + \alpha k_\nu\, k\cdot\epsilon + \beta \epsilon_\nu = 0$$

4 solutions

1) $\epsilon_\mu \propto k_\mu$ $(\epsilon \parallel k)$ (1 solution) $\qquad k^2 k_\nu + \alpha k^2 k_\nu + \beta k_\nu = 0$

$$\Rightarrow k^2 + \alpha k^2 + \beta = 0$$

$$k^2 = -\frac{\beta}{1+\alpha} \equiv \mu_s^2$$

2) $\epsilon\cdot k = 0$ (3 solutions)

$$(k^2 + \beta)\epsilon_\nu = 0 \Rightarrow k^2 = -\beta = -\mu^2$$

If we only want to describe a massive spin 1 state, take $\alpha = -1$ so that there is no solution for $\epsilon_\mu \propto k_\mu$ for $p \neq 0$..

$$\mathcal{L} = -\tfrac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A^\mu \partial_\nu A^\nu + \mu^2 A_\mu A^\mu\right]$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mu^2 A_\mu A^\mu$$

eq of motion: $\partial_\nu F^{\mu\nu} + \mu^2 A^\mu = 0$

Since $\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0 \Rightarrow \mu^2 \partial_\mu A^\mu = 0$ — this is a constraint which eliminates scalar $J=0$ piece.

Using this, can rewrite E of M as $\boxed{\Box A^\nu - \mu^2 A^\nu = 0 \,, \; \partial_\mu A^\mu = 0}$

solutions are $A_\mu = \epsilon_\mu e^{ikx} + \text{h.c.}$ with $k \cdot \epsilon = 0$, $k^2 = -\mu^2$.

Note: ϵ can be complex.

Three sol'ns $\epsilon_\mu^{(r)}$ $r = 1,2,3$ choose to be orthonormal:

$$\epsilon_\mu^{(r)} \epsilon^{\mu(s)} = \delta^{rs}$$

rest frame: $k^\mu = (\mu, 0, 0, 0)$

$\varepsilon^1_\mu = (0, 1, 0, 0)$

$\varepsilon^2_\mu = (0, 0, 1, 0)$

$\varepsilon^3_\mu = (0, 0, 0, 1)$

Note: $\sum_r \varepsilon^r_\mu \varepsilon^r_\nu = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$= g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2} \equiv P_{\mu\nu}$

Boost: in $\hat{z}$ direction

$k^\mu = (\omega, 0, 0, k) \qquad \omega^2 = k^2 + \mu^2$

$\varepsilon^1_\mu = (0, 1, 0, 0)$

$\varepsilon^2_\mu = (0, 0, 1, 0)$

$\varepsilon^3_\mu = \frac{1}{\mu}(k, 0, 0, \omega)$

still have $\varepsilon^r_\mu \varepsilon^{s\,\mu} = \delta_{rs}$

$\varepsilon^r_\mu \varepsilon^r_\nu = \frac{1}{\mu^2}\begin{pmatrix} k^2 & & & k\omega \\ & 1 & & \\ & & 1 & \\ k\omega & & & \omega^2 \end{pmatrix}$

$= g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$

$= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \frac{\begin{pmatrix} \omega^2 & k\omega \\ k\omega & k^2 \end{pmatrix}}{\mu^2}$

$= \cancel{-1 + \frac{\omega^2}{\mu^2}} \quad \omega$

$\frac{\omega^2}{\mu^2} - 1 = \frac{\mu^2 + k^2}{\mu^2} - 1 = \frac{k^2}{\mu^2}$

$\frac{k^2}{\mu^2} + 1 = \frac{\omega^2}{\mu^2}$

other bases ok, eg $\varepsilon^\pm = (\varepsilon^1 \pm i\varepsilon^2)/\sqrt{2}$

Still have $\varepsilon^{(r)}_\mu \varepsilon^{*(s)\,\mu} = +\delta_{rs}$

and $\sum_r \varepsilon^{(r)}_\mu \varepsilon^{*(r)}_\nu = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} = P_{\mu\nu}$

Note: $k^\mu P_{\mu\nu} = 0$

$P_{\mu\nu} = P_{\nu\mu}$

$\varepsilon^{(r)}_\mu$ = polarization vector $\quad \varepsilon^{1,2}_\mu \equiv$ transverse, $\quad \varepsilon^3_\mu$ = longitudinal.

(direction of $\vec{k}$)

Quantize

canonical momentum: $\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -(\partial^0 A^\mu - \partial^\mu A^0)$

$= +F^{\mu 0}$

so $\Pi^0 = 0$ $\quad \pi^i = F^{i0} = -E^i \leftarrow$ electric field.

Not a problem: recall, $\partial_\mu A^\mu = 0$ for $\mu^2 \neq 0$ $\Rightarrow$ can solve for A_0 given A_i $\Rightarrow$ so A_0 is not dynamical.

so A^i, F^{i0} completely specify system.

quantization: $[A_i(\vec{x},t), \pi^j(\vec{y},t)]$

$= [A_i(x,t), F^{j0}(\vec{y},t)] = i \underbrace{\delta_i{}^j}_{= g_i{}^j} \delta^3(\vec{x}-\vec{y})$

or $[A^i(x), F^{j0}(y,t)] = i g^{ij} \delta^3(\vec{x}-\vec{y}) = i\delta(\vec{x}-\vec{y})$

Expand in $a, a^\dagger$:

$$A_\mu = \sum_{r=1}^{3} \int d^3\tilde{k} \left[a_k^{(r)} \varepsilon_{k\mu}^{(r)} e^{ik\cdot x} + a_k^{\dagger(r)} \varepsilon_{k\mu}^{*r} e^{-ik x} \right]$$

find: $[a_k^{(r)}, a_{k'}^{\dagger(s)}] = (2\pi)^3 2\omega_k \delta^3(k-k')\delta_{rs}$

(5)

$$-\tfrac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$-\tfrac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu A_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \mu^2 A_\mu A^\mu$$

$$= -\tfrac{1}{2} A_\alpha \underbrace{\left(-\partial_\mu \partial^\mu g^{\alpha\beta} + \partial^\alpha \partial^\beta + \mu^2 g^{\alpha\beta}\right)} A_\beta$$

$$\left(-\partial_\mu \partial^\mu + \mu^2\right) g^{\alpha\beta} + \partial^\alpha \partial^\beta$$

$$\rightarrow \quad \left(k^2 + \mu^2\right) g^{\alpha\beta} - k^\alpha k^\beta$$

inverse $= \quad C_1 g_{\beta\gamma} + C_2 k_\beta k_\gamma = \Delta_{\beta\gamma}$

$$\text{set } \left((k^2+\mu^2) g^{\alpha\beta} - k^\alpha k^\beta\right) \Delta_{\beta\gamma} = g^\alpha{}_\gamma = \delta^\alpha{}_\gamma$$

$$\Rightarrow (k^2+\mu^2) C_1 g^\alpha{}_\gamma + C_2 (k^2+\mu^2) k^\alpha k_\gamma - C_1 k^\alpha k_\gamma - C_2 k^\alpha k_\gamma k^2 = g^\alpha{}_\gamma$$

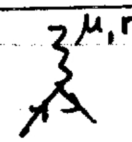
so $\quad C_2(k^2+\mu^2) - C_1 - C_2 k^2 = 0$

$$C_1(k^2+\mu^2) = 1$$

$$C_1 = \frac{1}{k^2+\mu^2} \qquad C_2(\mu^2) = +\frac{1}{k^2+\mu^2}$$

so $\quad \Delta_{\mu\nu} = \dfrac{1}{k^2+\mu^2}\left(g_{\mu\nu} + \dfrac{k_\mu k_\nu}{\mu^2}\right) = \dfrac{P_{\mu\nu}}{k^2+\mu^2}$

$$\boxed{\text{So propagator is} \quad \mu \overset{k}{\rightsquigarrow} \nu \quad -i\,\frac{P_{\mu\nu}}{k^2+\mu^2} \qquad P_{\mu\nu} = g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}}$$

external lines:  ϵ^{r*}_μ for incoming
ϵ^{r}_μ for outgoing
photons

Massless spin-1 bosons (photons)

Can we take $\mu \to 0$ limit?

consider eq of motion for massive A_μ couple to source J_μ:

$$\mathcal{L}_{int} = e A_\mu J^\mu$$

$$\Rightarrow \quad \partial_\nu F^{\mu\nu} + \mu^2 A^\mu + e J^\mu = 0$$

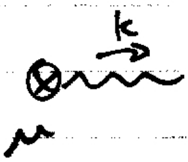
take ∂_μ: $\quad 0 + \mu^2 \partial_\mu A^\mu + e\,\partial_\mu J^\mu = 0$

$\mu^2 \to 0$ limit only possible if $\partial_\mu J^\mu = 0$.

Is conservation of J enough? What happens to $\frac{k_\mu k_\nu}{\mu^2}$ part of propagator?

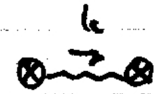
quantum theory of A_μ coupled to classical current J_μ

$$\tilde{J}_\mu(k) = \int d^4x \, e^{-ikx} J_\mu(x)$$



$-ie\tilde{J}(k)$ Feynman rule.

lowest order contribution of vac → vac amplitude in presence of J

$\langle 0 | S-1 | 0 \rangle_J =$ 

$\tilde{J}_\mu(k) \; J_\nu(-k)$

$$= -\frac{e^2}{2} \int \frac{d^4k}{(2\pi)^4} \, \tilde{J}_\mu(k) \, \frac{-i(g^{\mu\nu} + k^\mu k^\nu/\mu^2)}{k^2 + \mu^2 + i\epsilon} \, \tilde{J}_\nu(-k)$$

$$\partial_\mu J^\mu = 0 \Rightarrow k_\mu \tilde{J}^\mu(k) = 0$$

$$\text{so} = \frac{ie^2}{2} \int \frac{d^4k}{(2\pi)^4} \, \tilde{J}_\mu(+k) \, \tilde{J}^\mu(k) \, \frac{1}{k^2 + \mu^2 - i\epsilon}$$

⇒ The bad $k^\mu k^\nu / \mu^2$ term decouples if $\partial_\mu J^\mu = 0$.

left with $^\mu \sim\!\sim\, ^\nu = \dfrac{-ig^{\mu\nu}}{k^2 + i\epsilon}$ massless photon prop in Feynman gauge.

A closer look at the $\mu \to 0$ limit

$$\partial_\nu F^{\mu\nu} = e J^\nu$$

invariant under gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

If $\partial_\mu A^\mu \neq 0$, can always make gauge transformation

Where Λ satisfies $\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu$

so new field $A'_\mu = A_\mu + \partial_\mu \Lambda$ satisfies $\partial_\mu A^{\mu\prime} = 0$.

Called "Lorentz gauge"

this still leaves additional gauge freedom for the new function Λ satisfying $\partial_\mu \partial^\mu \Lambda = 0$.

Consequence of gauge invariance:

Before, with $\mu \neq 0$, we found

$k^\mu = (\omega, 0, 0, k)$

$\varepsilon^1 = (0, 1, 0, 0)$

$\varepsilon^2 = (0, 0, 1, 0)$

$\varepsilon^3 = (k, 0, 0, \omega)/\mu$

as $\mu \to 0$ $k^\mu \to k(1, 0, 0, 1)$

$\varepsilon^3_\nu \to k_\nu/\mu$

So 3rd solution is

$$A_\nu = \varepsilon^3_\nu e^{ikx} \to \frac{k_\nu}{\mu} e^{ikx}$$

or $A_\nu = -\frac{i}{\mu}\partial_\nu e^{ikx}$

Suppose I now shift $A_\nu \to A_\nu - \frac{i}{\mu}\partial_\nu e^{ikx}$

= gauge transformation with $\Lambda = -\frac{i}{\mu} e^{ikx}$

note that $\partial_\mu \partial^\mu \Lambda = k^2 \cdot \frac{i}{\mu} e^{ikx} = -i\mu e^{ikx} \xrightarrow[\mu \to 0]{} 0$

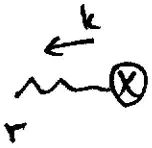
So as $\mu \to 0$ can (i) Shift away ε^3_ν solution
(ii) while maintaining $\partial_\mu A^\mu = 0$,
Which is what killed the scalar mode.

∴ massless photons only have two physical states.

$\varepsilon^1, \varepsilon^2$ or $\varepsilon^\pm = \frac{\varepsilon^1 \pm i\varepsilon^2}{\sqrt{2}}$ (transverse, or circular polarizations)

The longitudinal state decouples smoothly as $\mu \to 0$

Consider classical source again:



$i\mathcal{M}^{(r)} \propto \varepsilon^{(r)*}_\nu \cdot \tilde{J}^\nu(k)$

look at $\cancel{i\mathcal{M}^{(3)}}$ $i\mathcal{M}^{(3)}$: $k^\mu = (\omega, 0, 0, k)$,

$\omega = k\left(1 + \mathcal{O}\left(\frac{\mu^2}{k^2}\right)\right)$

Since the current is conserved, we also have

$$\left.\begin{aligned} k_\mu \tilde{J}^\mu &= 0 \\ &= -\omega \tilde{J}^0 + k \tilde{J}^3 \end{aligned}\right\} \Rightarrow \quad \tilde{J}^3 = \frac{\omega}{k}\tilde{J}^0 = \tilde{J}^0\left(1 + \mathcal{O}\left(\frac{\mu^2}{k^2}\right)\right)$$

$$\therefore \quad i\mathcal{M}^{(3)} \propto \varepsilon^{(3)}_\nu \tilde{J}^\nu = \frac{-k\tilde{J}^0 + \omega\tilde{J}^3}{\mu} = \tilde{J}^0\left(-\frac{k}{\mu} + \frac{\omega}{\mu} + \mathcal{O}\left(\frac{\mu^2}{k^2}\right)\right)$$

$$\propto \tilde{J}^0 \times \mathcal{O}\left(\frac{\mu}{k}\right)$$

$$\xrightarrow[\mu\to 0]{} 0 \quad \text{smoothly.}$$

Gauge theories

We now know how to quantize photons in a gauge where $\partial_\mu A^\mu = 0$.

Let us look at general $U(1)$ gauge theories. As we saw before, we can couple A_μ to fields through the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$

$\bar{\Psi}(iD_\mu\gamma^\mu - m)\Psi$ fermion

$-(D_\mu\phi)^*(D^\mu\phi) - m^2|\phi|^2$ scalar