Lorentz group for Physics 571 (Winter 2012)

sign convention corrected 1/6/12

Consider Lorentz transformations, for which the defining representation is 4-dimensional. This is the group of real matrices Λ which satisfy

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu} = \eta_{\alpha\beta} \ , \qquad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \ , \qquad \det\Lambda = 1 \ . \tag{1}$$

With this definition, the inner product between two 4-vectors, $v^{\mu}\eta_{\mu\nu}w^{\mu}$, is preserved under the Lorentz transformations $v \to \Lambda v$ and $w \to \Lambda w$, where Λ is a real, 4×4 matrix.

We can write

$$\Lambda = e^{i\theta_a X_a} \ , \qquad X_a = -i\frac{\partial\Lambda}{\partial\theta_a}\Big|_{\theta=0} \tag{2}$$

where the θ_a are the six real parameters for rotations and boosts, and the X_a are imaginary 4×4 matrices. Expanding that eq. (1) to linear order in the θ_a, one finds the X_a must satisfy

$$(X_a)_{\alpha\beta} + (X_a)_{\beta\alpha} = 0. \tag{3}$$

In other words, with both indices lowered, the X_a are antisymmetric. (But note that for the matrix $\Lambda^{\mu}{}_{\nu}$ we need to exponentiate $(X_a)^{\mu}{}_{\nu}$, with one index up and one down...these will **not** all be antisymmetric since, for example, if $M_{01} = -M_{10}$, then $M^{0}{}_{1} = +M^{1}{}_{0}$).

It is convenient to write the six linearly independent solutions of eq. (3) for the X_a as

$$(J_i)_{\mu\nu} = i\epsilon_{0i\mu\nu} \ , \qquad (K_i)_{\mu\nu} = -i\left(\eta_{\mu 0}\eta_{\nu i} - \eta_{\mu i}\eta_{\nu 0}\right) \tag{4}$$

where ϵ is the totally antisymmetric Levi-Civita symbol with $\epsilon_{0123} = -1$, while $(J_i)^{\mu}{}_{\nu} \equiv J_i$ and $(K_i)^{\mu}{}_{\nu} \equiv K_i$ with $i = 1, 2, 3$ are the generators for rotations and boosts respectively; e.g., J_3 generates rotations about the z axis, while K_3 generates boosts along the z axis where

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Note that J_i and K_i are not given by eq. (4), but have their first index raised. They are called "the four dimensional defining representation of the Lie algebra for the Lorentz group".

From these matrices you can find the abstract algebra that must be obeyed by the Lorentz generators in any representation:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \ , \quad [J_i, K_j] = i\epsilon_{ijk}K_k \ , \quad [K_i, K_j] = -i\epsilon_{ijk}J_k \ . \tag{6}$$

If we can find a d-dimensional matrix representation of this algebra (call the representation "R"), then a Lorentz transformation will be given by the d-dimensional matrix

$$D_R(\vec{\theta}, \vec{\omega})^{\mu}{}_{\nu} = \left(e^{i\vec{\theta}\cdot\vec{J} + \vec{\omega}\cdot\vec{K}}\right)^{\mu}{}_{\nu} \tag{7}$$

where $\vec{\theta}$ are the three real rotation angles and $\vec{\omega}$ are the three real boost parameters (The relation between the parameter $\vec{\omega}$ and the boost velocity $\vec{v}$ is $\hat{\omega}\tanh|\omega| = \vec{v}/c$). Since K_i is anti-hermitian, Λ will not be unitary. The goal is to find all the finite dimensional representations.

This algebra eq. (6) should look reminiscent of $SU(2)$. Next we define six Hermitian generators $\vec{A}$ and $\vec{B}$, with less physical meaning but a simpler algebra:

$$A_i \equiv \frac{1}{2}(J_i - iK_i) \qquad B_i \equiv \frac{1}{2}(J_i + iK_i) \tag{8}$$

From eq. (6) it follows that $\vec{A}$ and $\vec{B}$ satisfy

$$[A_i, A_j] = i\epsilon_{ijk}A_k \ , \quad [B_i, B_j] = i\epsilon_{ijk}B_k \ , \quad [A_i, B_j] = 0 \tag{9}$$

...in other words this is the group $SU(2) \times SU(2)$. Luckily we know all the irreducible representations of this group, as they are just labelled by two j quantum numbers:

$$R = (j_A, j_B) \tag{10}$$

a representation with dimension $(2j_A + 1)(2j_B + 1)$. You should think of where $A_i = a_i \otimes 1_B$ where a_i is a nontrivial $(2j_A + 1)$ matrix acting on the m_A indices, and 1_B is the trivial $(2j_B + 1)$ dimensional unit matrix acting on the m_B indices, and conversely $B_i = 1_A \otimes b_i$. The corresponding Lorentz transformation, from eq. (8) and eq. (7) is given by

$$D_{\{j_A, j_B\}} = e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{A}+i(\vec{\theta}-i\vec{\omega})\cdot\vec{B}} = e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{a}} \otimes e^{i(\vec{\theta}-i\vec{\omega})\cdot\vec{b}} \tag{11}$$

Note the interesting feature that

$$D^*_{\{j_A, j_B\}}(\vec{\theta}, \vec{\omega}) = e^{i(\vec{\theta}-i\vec{\omega})\cdot\vec{(-A^*)}+i(\vec{\theta}+i\vec{\omega})\cdot\vec{(-B^*)}} = D_{\{\bar{j}_B, \bar{j}_A\}} = D_{\{j_B, j_A\}} \tag{12}$$

where I used the facts that

(i) in general, if X_a is a representation R of a Lie algebra, then $-X_a^*$ is also a representation, and is called the conjugate representation (here denoted with a bar);

(ii) the group $SU(2)$ only has real representations, so j and $\bar{j}$ are the same (up to a similarity transformation);

(iii) since conjugation flipped the relative sign between θ and ω, the roles of A and B are flipped.

Thus if ψ transforms according to the (j_A, j_B) representation, ψ^* is in the (j_B, j_A) representation.

A related observation which is useful is that

$$D^\dagger_{\{j_A, j_B\}}(\vec{\theta}, \vec{\omega}) = e^{-i(\vec{\theta}-i\vec{\omega})\cdot\vec{A}-i(\vec{\theta}+i\vec{\omega})\cdot\vec{B}} = D^{-1}_{\{j_B, j_A\}}(\vec{\theta}, \vec{\omega}). \tag{13}$$

We now know all of the irreducible, finite dimensional representations of the Lorentz group. For example, assume we have a field ψ transforming as the $(\frac{1}{2}, 0)$ representation. Thus we have $\vec{a} = \frac{1}{2}\vec{\sigma}$ and $\vec{b} = 0$ and

$$D_L \equiv D_{(\frac{1}{2},0)}(\vec{\theta}, \vec{\omega}) = e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{\sigma}/2} \ . \tag{14}$$

A rotation by angle θ_3 about the z axis corresponds to the matrix

$$D_{(\frac{1}{2},0)}(\theta_3) = e^{i\theta_3\sigma_3/2} = \begin{pmatrix} e^{i\theta_3/2} & 0 \\ 0 & e^{-i\theta_3/2} \end{pmatrix} \ . \tag{15}$$

A boost in the z direction with velocity parameter ω_3 corresponds to

$$D_{(\frac{1}{2},0)}(\omega_3) = e^{-\omega_3\sigma_3/2} = \begin{pmatrix} e^{-\omega_3/2} & 0 \\ 0 & e^{\omega_3/2} \end{pmatrix} \tag{16}$$

$\hat{z}$ rotations multiply the components of ψ by phases (a factor of -1 for a rotation by 2π), while a boost makes the upper component exponentially large, and the lower component exponentially small.

Similarly, if χ transforms according to the $(0, \frac{1}{2})$ representation, then $\chi \to D_R \chi$ where

$$D_R \equiv D_{(0,\frac{1}{2})} = e^{i(\vec{\theta} - i\vec{\omega})\cdot\vec{\sigma}/2} \ . \tag{17}$$

Rotations are the same as for ψ, but boosts have $\omega \to -\omega$.