QFT 570 Homework 6

Lorentz Symmetry

a.

Quickly following the hint to show that the property $\sigma_2\sigma_i^T\sigma_2 = -\sigma_i$ is true through application of 1.) the pauli matrices' anti-commutation property $\{\sigma_i, \sigma_j\} = 0$ and 2.) the fact that they all square to the identity:

1. The transpose of σ_1 is itself:
$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0 \longrightarrow \sigma_2\sigma_1^T\sigma_2 = -\sigma_1$$
2. The transpose of σ_2 is negative itself:
$$\sigma_2\sigma_2^T\sigma_2 = -\sigma_2$$
3. The transpose of σ_3 is itself:
$$\sigma_3\sigma_2 + \sigma_2\sigma_3 = 0 \longrightarrow \sigma_2\sigma_3^T\sigma_2 = -\sigma_3$$

Thus,

$$\sigma_2\sigma_i^T\sigma_2 = -\sigma_i$$

So we have a transformation of ϕ defined as

$$\phi_{is} \to \left[e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{A}}\right]_{ij}\left[e^{i(\vec{\theta}-i\vec{\omega})\cdot\vec{B}}\right]_{st}\phi_{jt} \equiv D^{j_A}_{ij}(\vec{\theta},\vec{\omega})D^{j_B}_{st}(\vec{\theta},\vec{\omega})\phi_{jt} \tag{1}$$

To write this transformation in matrix notation, we must order the indices leading to the necessity of a transpose operation on the second D.

$$\phi = \chi\sigma_2 \to D^{j_A}(\vec{\theta},\vec{\omega})\phi D^{j_B}(\vec{\theta},\vec{\omega})^T \tag{2}$$
$$= D^{j_A}(\vec{\theta},\vec{\omega})\chi\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1}D^{j_B}(\vec{\theta},\vec{\omega})\sigma_2 D^{j_B}(\vec{\theta},\vec{\omega})^T \tag{3}$$
$$= D^{j_A}(\vec{\theta},\vec{\omega})\chi\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1}D^{j_B}(\vec{\theta},\vec{\omega})\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1}\sigma_2 \tag{4}$$
$$= D^{j_A}(\vec{\theta},\vec{\omega})\chi\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1}\sigma_2 \tag{5}$$

$$\chi \to D^{j_A}(\vec{\theta},\vec{\omega})\chi\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1} \tag{6}$$

In the first line, I have inserted $\mathbf{1}$ between χ and σ_2 in the definition of $\phi = \chi\sigma_2$. In the next line, I use the equation in the hint to state $\sigma_2\sigma_i^T = -\sigma_i\sigma_2$ and thus pulling σ_2 to the right causes the transposed transformation to become the inverse (ok, the added identity was completely unnecessary...). More rigorously, this may be shown by expanding D's exponential form. From here, we may simply right-multiply by σ_2 on both sides to isolate the transformation of χ.

b.

If χ transforms as

$$\chi \to D^{j_A}(\vec{\theta},\vec{\omega})\chi\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1}$$

we can make a statement about the transformation property of the determinant by applying the fact $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. Taking the determinant of both sides:

$$\det\chi \to \det\left[D^{j_A}(\vec{\theta},\vec{\omega})\right]\det\left[\left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1}\right]\det\chi \tag{7}$$

Imagine that the matrices representing the Lorentz transformation can be diagonalized. The determinant is simply the product of eigenvalues. The determinant of the inverse is the product of eigenvalues^{-1}. Eigenvalue-by-eigenvalue, this product will be unity and thus the determinant of χ is invariant under Lorentz transformation.

c.

Writing χ as $p_\mu\sigma^\mu$, χ is written as:

$$\chi = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \qquad \det\chi = p_0^2 - p_1^2 - p_2^2 - p_3^2 \tag{8}$$

When expressed in terms of a 4-vector notation, the determinant is suggestive of the dot product $p \cdot p$ in a mostly-minus metric and is thus not surprising that it is a lorentz invariant.

d. and e.

Under rotations where $\vec{\theta} = 0$ or boosts where $\vec{\omega} = 0$, the $\left(\frac{1}{2},\frac{1}{2}\right)$ transformation matrices simplify.

$$D^{j_A}_{ij}(\vec{\theta},0) = \left[e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\right]_{ij} \qquad D^{j_B}_{ij}(\vec{\theta},0) = \left[e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\right]_{ij}$$

$$D^{j_A}_{ij}(0,\vec{\omega}) = \left[e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\sigma}}\right]_{ij} \qquad D^{j_B}_{ij}(0,\vec{\omega}) = \left[e^{\frac{1}{2}\vec{\theta}\cdot\vec{\sigma}}\right]_{ij}$$

We can show that χ's transformation properties may be summarized into the 4-vector nature of p_μ in general by using Mathematica and looking at the following equalities:

$$\chi \to D^{j_A}(\vec{\theta},0)\chi\left(D^{j_B}(\vec{\theta},0)\right)^{-1} \stackrel{?}{=} \left[e^{i\vec{\theta}\vec{J}}\right]_\mu^{\ \nu} p_\nu\sigma^\mu \qquad \text{and} \qquad \chi \to D^{j_A}(0,\vec{\omega})\chi\left(D^{j_B}(0,\vec{\omega})\right)^{-1} \stackrel{?}{=} \left[e^{i\vec{\omega}\cdot(-\vec{K})}\right]_\mu^{\ \nu} p_\nu\sigma^\mu$$

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J1 = SparseArray[{{3, 4} → -i, {4, 3} → i}];
J2 = SparseArray[{{2, 4} → i, {4, 2} → -i}];
J3 = SparseArray[{{2, 3} → -i, {3, 2} → i, {4, 4} → 0}];
Jvec = {J1, J2, J3};
pmu = {p0, p1, p2, p3};
σmu = {IdentityMatrix[2], PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
θvec = {θ1, θ2, θ3};

Simplify[Simplify[ExpToTrig[(MatrixExp[i θvec.Jvec].pmu).σmu]] ==
  Simplify[ExpToTrig[MatrixExp[i/2 θvec.σmu[[2 ;; 4]]].(pmu.σmu).
    MatrixExp[-i/2 θvec.σmu[[2 ;; 4]]]]]]

True
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K1 = SparseArray[{{1, 2} → -i, {2, 1} → -i, {4, 4} → 0}];
K2 = SparseArray[{{1, 3} → -i, {3, 1} → -i, {4, 4} → 0}];
K3 = SparseArray[{{1, 4} → -i, {4, 1} → -i}];
Kvec = -{K1, K2, K3};
pmu = {p0, p1, p2, p3};
σmu = {IdentityMatrix[2], PauliMatrix[1], PauliMatrix[2], PauliMatrix[3]};
ωvec = {ω1, ω2, ω3};

Simplify[
 Simplify[ExpToTrig[MatrixExp[-1/2 ωvec.σmu[[2 ;; 4]]].(pmu.σmu).
    MatrixExp[-1/2 ωvec.σmu[[2 ;; 4]]]]] ==
  Simplify[ExpToTrig[(MatrixExp[i ωvec.Kvec].pmu).σmu]]]

True
```