QFT 570 Homework 5

heavy-ϕ theory

The Lagrangian for the following contains kinetic and mass terms for two types of fields, φ and χ, with a 3-point φ interaction and a 2-1 χ-φ interaction.

$$\mathcal{L} = -\frac{1}{2}Z_\varphi \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}Z_m m^2 \varphi^2 + Y\varphi - \frac{1}{2}Z_\chi \partial^\mu \chi \partial_\mu \chi - \frac{1}{2}Z_M M^2 \chi^2 + \frac{1}{6}Z_g g \tilde{\mu}^{\frac{\epsilon}{2}} \varphi^3 + \frac{1}{2}Z_h h \tilde{\mu}^{\frac{\epsilon}{2}} \varphi \chi^2 \tag{1}$$

a.

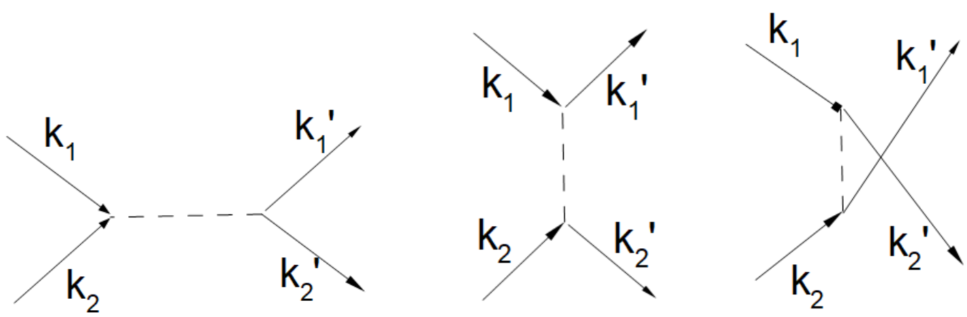

Figure 1: Tree-level diagrams contributing to $\chi\chi \to \chi\chi$ scattering. The solid lines represent external χs while the dashed lines represent internal φs.

There are three diagrams contributing to $\chi\chi$ scattering at tree level. They are analogous to the s, t, and u, channel diagrams we would see in scalar χ^3 but here contain internal φs. The amplitude for such scattering may be built similarly by reading the feynman rules from the Lagrangian as:

$$i\mathcal{T} = (iZ_h h)^2 \left[\frac{-i}{m^2 - s - i\epsilon} + \frac{-i}{m^2 - t - i\epsilon} + \frac{-i}{m^2 - u - i\epsilon}\right] \tag{2}$$

$$= ih^2 \left[\frac{1}{m^2 - s - i\epsilon} + \frac{1}{m^2 - t - i\epsilon} + \frac{1}{m^2 - u - i\epsilon}\right] \tag{3}$$

where I have set the lowest-order value of Z_h equal to unity and have used the usual definitions of the mandelstam variables for the mostly-plus metric

$$-s = (k_1 + k_2)^2 \qquad -t = \left(k_1 - k_1'\right)^2 \qquad -u = \left(k_1 - k_2'\right)^2$$

In the center of mass frame, $\vec{k}_1 = -\vec{k}_2$ and $\vec{k}_1' = -\vec{k}_2'$.

b.

Here, we expand the above amplitude to second order in the external χ momentum to focus on low-energy interactions of χs. Ignoring the $i\epsilon$s for the moment:

$$i\mathcal{T} = ih^2 \frac{1}{m^2} \left[\frac{1}{1 - \frac{s}{m^2}} + \frac{1}{1 - \frac{t}{m^2}} + \frac{1}{1 - \frac{u}{m^2}}\right] \tag{4}$$

$$= ih^2 \frac{1}{m^2} \left[\left(1 + \frac{s}{m^2} + \cdots\right) + \left(1 + \frac{t}{m^2} + \cdots\right) + \left(1 + \frac{u}{m^2} + \cdots\right)\right] \tag{5}$$

$$= ih^2 \frac{1}{m^2} \left[3 + \frac{4M^2}{m^2} + \mathcal{O}\left(k^4\right) \right] \tag{6}$$

where I have used the condition $s + t + u = 4M^2$. As we intend to be in the limit $M \ll m$, the resulting expansion in $\frac{M}{m}$ is expected.

c.

The proposed effective theory retains the kinetic and mass terms for only the χ field and interaction terms $\sim \chi^4$.

$$\mathcal{L}_{EFT} = -\frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{1}{2} M^2 \chi^2 - \frac{a}{m^2}\frac{\chi^4}{4!} - \frac{bM^2}{m^4}\frac{\chi^4}{4!} - \frac{c}{m^4}\frac{\left(\partial_\mu \chi \partial^\mu \chi\right)\chi^2}{4} + \cdots$$

The tree level scattering in the effective theory has contributions from diagrams with each 4-point vertex.

$$i\mathcal{T}_{EFT} = -i\frac{a}{m^2} - i\frac{bM^2}{m^4} + \sum_{i,j\ i \neq j} i\frac{c}{m^4}(p_i \cdot p_j) \tag{7}$$

where the 2-derivative vertex has been evaluated with all momenta incoming/outgoing so that the negative signs associated with the momentum orientations cancel and we are left only with the $(-i)^2$ minus sign in momentum space (schematically, $\partial\partial \to -p_i \cdot p_j$).

d.

By comparing the results from the two theories at a renormalization scale of $\mu = m$, we will find the matching conditions on the effective coupling constants a, b, and c.

$$a = -3h^2$$

$$b = -4h^2$$

$$c = 0$$

e.

This operator interpreted in momentum space gives a multiplicative factor of $-k^2$ on the 4-χ vertex where k is the 4-momentum of one of the external χs. Because our external legs are placed on shell, this factor of $k^2 = -M^2$ making this term redundant in light of the included b term.

2D SHO

$$\mathcal{H} = \frac{1}{2}\left(p_i p_i + \omega^2 x_i x_i\right) = \omega\left(a_i^\dagger a_i + 1\right) \tag{8}$$

First, we need to reinsert the factors of ω in the ladder operators:

$$a_i = \sqrt{\frac{\omega}{2}}\left(x_i + \frac{i}{\omega}p_i\right) \qquad a_i^\dagger = \sqrt{\frac{\omega}{2}}\left(x_i - \frac{i}{\omega}p_i\right) \tag{9}$$

now, the 2-dimensional Hamiltonian in terms of the number operator is indeed the expected Hamiltonian in x and p.

$$\omega\left(a_i^\dagger a_i + 1\right) = \omega\left(\frac{\omega}{2}\left(x_i - \frac{i}{\omega}p_i\right)\left(x_i + \frac{i}{\omega}p_i\right) + 1\right) \tag{10}$$

$$= \omega \left(\frac{\omega}{2} \left(x_i x_i + \frac{1}{\omega^2} p_i p_i + \frac{i}{\omega} \underbrace{[x_i, p_i]}_{i\delta_{ii}} \right) + 1 \right) \tag{11}$$

$$= \frac{1}{2} \left(p_i p_i + \omega^2 x_i x_i \right) \tag{12}$$

These ladder operators still satisfy

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger \right] = 0 \qquad \left[a_i, a_j^\dagger \right] = \delta_{ij} \tag{13}$$

a.

Degeneracy of the n^{th} energy level occurs when multiple n_1 and n_2 occupations result in the same value of $n_1 + n_2 = n$. For example, the level $n = 2$ has degeneracies from $|n_1, n_2\rangle$ states: $|2,0\rangle, |1,1\rangle,$ and $|0,2\rangle$. For any particular value of n, once n_1 is chosen, n_2 is fixed. For each value of n, there are $n+1$ possible values of n_1 that satisfy the requirement of integer positivity for both n_1 and n_2. This means the degeneracy for the energy state E_n is $n+1$. Examples for a sanity check:

$$n = 0 \quad E = 1\omega \qquad |0,0\rangle \tag{14}$$

$$n = 1 \quad E = 2\omega \qquad |1,0\rangle \quad |0,1\rangle \tag{15}$$

$$n = 2 \quad E = 3\omega \qquad |2,0\rangle \quad |1,1\rangle \quad |0,2\rangle \tag{16}$$

$$n = 3 \quad E = 4\omega \qquad |3,0\rangle \quad |2,1\rangle \quad |1,2\rangle \quad |0,3\rangle \tag{17}$$

$$n = 4 \quad E = 5\omega \qquad |4,0\rangle \quad |3,1\rangle \quad |2,2\rangle \quad |1,3\rangle \quad |0,4\rangle \tag{18}$$

b.

If David defines the operators $O_A = a^\dagger A a$ and $O_B = a^\dagger B a$, then their commutator may be evaluated as

$$[O_A, O_B] = \left[a_i^\dagger A_{ij} a_j, a_k^\dagger B_{k\ell} a_\ell \right] \tag{19}$$

$$= a_i^\dagger \left[A_{ij} a_j, a_k^\dagger \right] B_{k\ell} a_\ell + a_k^\dagger \left[a_i^\dagger, B_{k\ell} a_\ell \right] A_{ij} a_j \tag{20}$$

$$= a_i^\dagger A_{ij} \delta_{jk} B_{k\ell} a_\ell - a_k^\dagger B_{k\ell} \delta_{i\ell} A_{ij} a_j \tag{21}$$

$$= a_i^\dagger A_{ik} B_{k\ell} a_\ell - a_k^\dagger B_{ki} A_{ij} a_j \tag{22}$$

$$= a_i^\dagger \left[A, B \right]_{i\ell} a_\ell \tag{23}$$

c.

Show that the operators Q_α are associated with a conserved vector-like quantity.

$$Q_\alpha = \frac{1}{2} a_i^\dagger \sigma_{ij}^\alpha a_j \tag{24}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q_1 = \frac{1}{2} \left(a_1^\dagger a_2 + a_2^\dagger a_1 \right) \tag{25}$$

$$Q_2 = \frac{i}{2} \left(a_2^\dagger a_1 - a_1^\dagger a_2 \right) \tag{26}$$

$$Q_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \tag{27}$$

Now the desired commutators are all in terms of ladder operators whose commutators we know.

$$\begin{aligned}
[Q_1, H] &= \left[\frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\right] &(28)\\
&= \frac{\omega}{2}\left(\left[a_1^\dagger a_2, a_1^\dagger a_1\right] + \left[a_1^\dagger a_2, a_2^\dagger a_2\right] + \left[a_2^\dagger a_1, a_1^\dagger a_1\right] + \left[a_2^\dagger a_1, a_2^\dagger a_2\right]\right) &(29)\\
&= \frac{\omega}{2}\left(a_1^\dagger \left[a_1^\dagger, a_1\right] a_2 + a_1^\dagger \left[a_2, a_2^\dagger\right] a_2 + a_2^\dagger \left[a_1, a_1^\dagger\right] a_1 + a_2^\dagger \left[a_2^\dagger, a_2\right] a_1\right) &(30)\\
&= \frac{\omega}{2}\left(-a_1^\dagger a_2 + a_1^\dagger a_2 + a_2^\dagger a_1 - a_2^\dagger a_1\right) &(31)\\
&= 0 &(32)
\end{aligned}$$

$$[Q_2, H] = \left[\frac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right), \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\right] = 0 \tag{33}$$

which can be determined from the evaluation of the Q_1 commutator by noting that the terms $a_2^\dagger a_1$ and $a_1^\dagger a_2$ individually cancel so changing the sign of one of them in the operator does not changes its commutation with the Hamiltonian.

$$\begin{aligned}
[Q_3, H] &= \left[\frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \omega\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] &(34)\\
&= \frac{\omega}{2}\left(\left[a_1^\dagger a_1, a_1^\dagger a_1\right] - \left[a_2^\dagger a_2, a_2^\dagger a_2\right]\right) &(35)\\
&= 0 &(36)
\end{aligned}$$

The last property of these charge operators we deduce from the known pauli relation

$$[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$$

(It may be helpful to think of the combination $\sigma_{ij}a_j$ simply as a mixed vector of a_1 and a_2).

$$\begin{aligned}
[Q_\alpha, Q_\beta] &= \frac{1}{4}\left[a_i^\dagger \sigma_{ij}^\alpha a_j, a_k^\dagger \sigma_{k\ell}^\beta a_\ell\right] &(37)\\
&= \frac{1}{4}\left(a_k^\dagger \left[a_i^\dagger, \sigma_{k\ell}^\beta a_\ell\right] \sigma_{ij}^\alpha a_j + a_i^\dagger \left[\sigma_{ij}^\alpha a_j, a_k^\dagger\right] \sigma_{k\ell}^\beta a_\ell\right) &(38)\\
&= \frac{1}{4}\left(-a_k^\dagger \sigma_{k\ell}^\beta \delta_{i\ell} \sigma_{ij}^\alpha a_j + a_i^\dagger \sigma_{ij}^\alpha \delta_{jk} \sigma_{k\ell}^\beta a_\ell\right) &(39)\\
&= \frac{1}{4}\left(-a_k^\dagger \sigma_{k\ell}^\beta \sigma_{\ell j}^\alpha a_j + a_k^\dagger \sigma_{k\ell}^\alpha \sigma_{\ell j}^\beta a_j\right) &(40)\\
&= \frac{1}{4} a_k^\dagger \left[\sigma^\alpha, \sigma^\beta\right]_{kj} a_j &(41)\\
&= \frac{1}{4} a_k^\dagger 2i\epsilon_{\alpha\beta\gamma}\sigma_{kj}^\gamma a_j &(42)\\
&= i\epsilon_{\alpha\beta\gamma} Q_\gamma &(43)
\end{aligned}$$

c. more simply using results of part b.

The commutator with the Hamiltonian can be quickly evaluated by mapping the form of the Hamiltonian into the form of operators O_A or O_B.

$$H = \omega\left(a_i^\dagger a_i + 1\right) = \omega\left(a^\dagger \mathbb{I} a + 1\right) \tag{44}$$

$$[Q_\alpha, H] = \left[Q_\alpha, a^\dagger \mathbb{I} a\right] = \frac{1}{2} a^\dagger \left[\sigma^\alpha, \mathbb{I}\right] a = 0 \tag{45}$$

$$\begin{aligned}
[Q_\alpha, Q_\beta] &= \frac{1}{4} a^\dagger \left[\sigma^\alpha, \sigma^\beta\right] a \\
&= \frac{i\epsilon_{\alpha\beta\gamma}}{2} a^\dagger \sigma^\gamma a \\
&= i\epsilon_{\alpha\beta\gamma} Q_\gamma
\end{aligned} \tag{46}$$

d.

Some more commutators!

$$\left[Q^2, H\right] = [Q_\alpha Q_\alpha, H] = Q_\alpha [Q_\alpha, H] + [Q_\alpha, H] Q_\alpha = 0 \tag{47}$$

We have already shown that Q_3 commutes with the Hamiltonian.

$$[Q_\alpha Q_\alpha, Q_3] = Q_\alpha [Q_\alpha, Q_3] + [Q_\alpha, Q_3] Q_\alpha \tag{48}$$
$$= Q_\alpha i\epsilon_{\alpha 3\beta} Q_\beta + i\epsilon_{\alpha 3\beta} Q_\beta Q_\alpha \tag{49}$$
$$= i\epsilon_{\alpha 3\beta} \{Q_\alpha, Q_\beta\} \tag{50}$$
$$= 0 \qquad \text{(by symmetry)} \tag{51}$$

Thus, Q^2 and Q_3 share a simultaneous eigenbasis with the energy eigenstates.

e.

Using the operators $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$, we rewrite Q^2 and Q_3 as follows:

$$Q_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) = \frac{1}{2}(N_1 - N_2) \tag{52}$$

$$Q^2 = Q_\alpha Q_\alpha = \frac{1}{4}\left(a_i^\dagger \sigma_{ij}^\alpha a_j a_k^\dagger \sigma_{k\ell}^\alpha a_\ell\right) \tag{53}$$
$$= \frac{1}{2} a_i^\dagger a_j a_k^\dagger a_\ell \left(\delta_{i\ell}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{k\ell}\right) \tag{54}$$
$$= \frac{1}{2}\left(a_i^\dagger a_k a_k^\dagger a_i - \frac{1}{2} a_i^\dagger a_i a_k^\dagger a_k\right) \tag{55}$$
$$= \frac{1}{2}\left(a_i^\dagger a_k \left(\left[a_k^\dagger, a_i\right] + a_i a_k^\dagger\right) - \frac{1}{2} a_i^\dagger a_i a_k^\dagger a_k\right) \tag{56}$$
$$= \frac{1}{2}\left(-a_i^\dagger a_k \delta_{ik} + a_i^\dagger a_i a_k a_k^\dagger - \frac{1}{2} a_i^\dagger a_i a_k^\dagger a_k\right) \tag{57}$$
$$= \frac{1}{2}\left(-a_k^\dagger a_k + a_i^\dagger a_i \left(\delta_{kk} + a_k^\dagger a_k\right) - \frac{1}{2} a_i^\dagger a_i a_k^\dagger a_k\right) \tag{58}$$
$$= \frac{1}{2}\left(a_i^\dagger a_i + \frac{1}{2} a_i^\dagger a_i a_k^\dagger a_k\right) \tag{59}$$
$$= \frac{1}{2} a_i^\dagger a_i \left(1 + \frac{1}{2} a_k^\dagger a_k\right) \tag{60}$$
$$= \frac{1}{4} a_i^\dagger a_i \left(2 + a_k^\dagger a_k\right) \tag{61}$$

$$= \frac{1}{4}(N_1 + N_2)(N_1 + N_2 + 2) \tag{62}$$

f.

We can determine the values of j and m by evaluating matrix elements in either of the simultaneous eigenbases. We may specify eigenstates of Q^2 by either the values of n_1, n_2 or the values of j, m. Regardless of what we choose to call the state, the associated eigenvalues must be equal.

$$Q^2|n_1, n_2\rangle = Q^2|j, m\rangle \tag{63}$$

$$\frac{1}{4}(N_1 + N_2)(N_1 + N_2 + 2)|n_1, n_2\rangle = Q^2|j, m\rangle \tag{64}$$

$$\frac{n_1 + n_2}{2}\frac{n_1 + n_2 + 2}{2}|n_1, n_2\rangle = j(j+1)|j, m\rangle \tag{65}$$

$$\Rightarrow j = \frac{(n_1 + n_2)}{2} \tag{66}$$

and for Q_3,

$$Q_3|n_1, n_2\rangle = Q_3|j, m\rangle \tag{67}$$

$$\frac{1}{2}(N_1 - N_2)|n_1, n_2\rangle = Q_3|j, m\rangle \tag{68}$$

$$\frac{1}{2}(n_1 - n_2)|n_1, n_2\rangle = m|j, m\rangle \tag{69}$$

$$\Rightarrow m = \frac{1}{2}(n_1 - n_2) \tag{70}$$

We can then write the energy E_n in terms of j and m as

$$E_n = \omega(n_1 + n_2 + 1) = \omega(2j + 1) \tag{71}$$

The possible values of j and m:

$$0 \leq j \qquad\qquad -j \leq m \leq j$$

Translated to constraints on n_1 and n_2, these possible values tell us that (j) neither n_1 nor n_2 are negative and (m) n_1 and n_2 can be separated maximally by their sum, n, as can be seen in the examples explicitly written above.

g.

Now, we have

$$[H, L_z] = 0 \qquad\qquad L_z = (x_1p_2 - x_2p_1)$$

There are two paths forward: 1.) use the expressions of x and p in terms of ladder operators to identify the structure of L_z with those of the Qs or 2.) find the matrix $\mathbf{X}$ satisfying the following relation:

$$\frac{\omega}{2}\begin{pmatrix}(x_1 - \frac{i}{\omega}p_1) & (x_2 - \frac{i}{\omega}p_2)\end{pmatrix}\mathbf{X}\begin{pmatrix}(x_1 + \frac{i}{\omega}p_1) \\ (x_2 + \frac{i}{\omega}p_2)\end{pmatrix} = x_1p_2 - x_2p_1$$

Clearly, diagonal elements of $\mathbf{X}$ will be 0. The off-diagonal elements must satisfy

$$x_1p_2 - x_2p_1 = \frac{\omega}{2}\left(\left(x_1 - \frac{i}{\omega}p_1\right)\mathbf{X}_{12}\left(x_2 + \frac{i}{\omega}p_2\right) + \left(x_2 - \frac{i}{\omega}p_2\right)\mathbf{X}_{21}\left(x_1 + \frac{i}{\omega}p_1\right)\right) \tag{72}$$

$$= \frac{\omega}{2}\left(\mathbf{X}_{12}\left(x_1x_2 + \frac{1}{\omega^2}p_1p_2 + \frac{i}{\omega}x_1p_2 - \frac{i}{\omega}p_1x_2\right) + \mathbf{X}_{21}\left(x_1x_2 + \frac{1}{\omega^2}p_1p_2 + \frac{i}{\omega}x_2p_1 - \frac{i}{\omega}x_1p_2\right)\right) \tag{73}$$

If we next set $\mathbf{X}_{12} = -\mathbf{X}_{21}$

$$x_1p_2 - x_2p_1 = \frac{\omega}{2}\mathbf{X}_{12}\frac{i}{\omega}\left(x_1p_2 - p_1x_2 - x_2p_1 + x_1p_2\right) \tag{74}$$

$$= i\mathbf{X}_{12}\left(x_1p_2 - x_2p_1\right)\mathbf{X}_{12} = -i \tag{75}$$

Thus, $\mathbf{X}$ is the y-pauli matrix

$$\mathbf{X} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{76}$$

so that $L_z = a_i^\dagger \sigma_{ij}^2 a_j = 2Q_2$. The vanishing commutator $[L_z, H]$ tell us that the two operators may share a simultaneous eigenbasis so that the energy eigenstates $|n\rangle$ are eigenstates of L_z. However, the eigenstates $|n_1, n_2\rangle$ are not eigenstates of L_z as can be seen by simply acting Q_2 on the state $|n_1, n_2\rangle$. The $|j, m\rangle$ eigenstates are not eigenstates of L_z as these are eigenstates of Q_3 and $[Q_3, Q_2] \neq 0$. The eigenvalues of L_z in the n^{th} energy level may be related to those of Q_3. As Q_3 and Q_2 are rotated versions of the same projection operator, their range of eigenvalues will be the same. Thus, we can expect eigenvalues of $L_z = 2Q_2$ to span between $\pm 2j$ or between $\pm n$.

Srednicki 22.1

This problem asks us to show

$$[\varphi_a, Q] = i\delta\varphi_a \tag{77}$$

where Q is the Noether charge associated with the Noether current:

$$j^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial\left(\partial_\mu \varphi_a(x)\right)}\delta\varphi_a(x) \tag{78}$$

$$[\varphi_a(x), Q] = \int \mathrm{d}^3x' \left[\varphi_a(x), \frac{\partial \mathcal{L}(x')}{\partial\left(\partial_0 \varphi_b(x')\right)}\delta\varphi_b(x')\right] \tag{79}$$

$$= \int \mathrm{d}^3x' \left[\varphi_a(\mathbf{x}, t), \Pi_b(\mathbf{x}', t)\delta\varphi_b(x)\right] \tag{80}$$

$$= \int \mathrm{d}^3x' \left[\varphi_a(\mathbf{x}, t), \Pi_b(\mathbf{x}', t)\right]\delta\varphi_b(x') \tag{81}$$

$$= \int \mathrm{d}^3x' i\delta^3(\mathbf{x} - \mathbf{x}')\delta_{ab}\delta\varphi_b(x') \tag{82}$$

$$= i\delta\varphi_a(x) \tag{83}$$

where I used the definition of the conjugate momentum to replace half of the 0^{th} component expression of the Noether current and took the assumption that perturbations of the field are time independent to be justification for assuming $\delta\varphi_b(x')$ commutes with $\varphi_a(x)$ and subsequent commutators are equal-time (though the charge is explicitly conserved, so the time at which Q is evaluated should be inconsequential).

An alternate approach is to expand the charge and field down to the level of creation and annihilation operators using Eq.(22.18-9) and the commutators as evaluated in HW1.