QFT 570 Homework 4

Srednicki 11.2: Compton Scattering $e^- \gamma \to e^- \gamma$

a.

Mandelstam variables s and u in terms of the initial and final photon energies ω and ω':

$$s = -\left(k_1 + k_2\right)^2 \qquad u = -\left(k_1 - k_2'\right)^2$$

in terms of electron/photon labels:

$$s = -\left(k_e + k_\gamma\right)^2 \qquad u = -\left(k_e - k_\gamma'\right)^2 = -\left(k_\gamma - k_e'\right)^2$$

The initial four vectors in the fixed target frame:

$$k_e = \left(m_e, \vec{0}\right) \qquad k_\gamma = \left(\omega, \vec{p}_\gamma\right)$$

$$\begin{aligned} s &= -\left(k_e + k_\gamma\right)^2 && (1) \\ &= -k_e^2 - k_\gamma^2 - 2k_e \cdot k_\gamma && (2) \\ &= m_e^2 + 0 + 2m_e\omega && (3) \\ &= m_e(m_e + 2\omega) && (4) \end{aligned}$$

$$\begin{aligned} u &= -\left(k_e - k_\gamma'\right)^2 && (5) \\ &= -k_e^2 - \left(k_\gamma'\right)^2 + 2k_e \cdot k_\gamma' && (6) \\ &= m_e^2 + 0 - 2m_e\omega' && (7) \\ &= m_e(m_e - 2\omega') && (8) \end{aligned}$$

b.

The scattering angle θ_{FT} between the initial and final photon 3-momenta may be encountered through the Mandelstam t. Having already calculated s and u, we get t through the constraint

$$s + t + u = \sum_i m_i^2 = 2m_e^2$$

$$\begin{aligned} t &= 2m_e^2 - s - u && (9) \\ &= 2m_e^2 - m_e\left(m_e + 2\omega\right) - m_e(m_e - 2\omega') && (10) \\ &= 2m_e(\omega' - \omega) = -\left(k_\gamma - k_\gamma'\right)^2 && (11) \\ &\qquad\qquad\qquad\quad = 2k_\gamma \cdot k_\gamma' && (12) \\ &\qquad\qquad\qquad\quad = -2\omega\omega' + 2|\vec{k}_\gamma||\vec{k}_\gamma'| \cos\theta_{FT} && (13) \\ &\qquad\qquad\qquad\quad = -2\omega\omega' + 2\omega\omega' \cos\theta_{FT} && (14) \end{aligned}$$

$$\cos\theta_{FT} = \frac{m_e(\omega' - \omega)}{\omega\omega'} + 1 \quad \Rightarrow \quad \theta_{FT} = \arccos\left[1 + \frac{m_e(\omega' - \omega)}{\omega\omega'}\right] \qquad (15)$$

c.

We have two options: 1.) Recall from Eq. 11.34 the expression of the differential scattering cross section for an arbitrary frame and specify to the fixed-target frame or 2.) go back to Eq. 11.22 and recalculate $dLIPS_2(k_e + k_\gamma)$. I choose to do the second where Eq. 11.22 with the substitution of Eq. 11.9 is

$$\mathrm{d}\sigma = \frac{1}{4m_e\omega}|\mathcal{T}|^2 dLIPS_2(k_e + k_\gamma) \tag{16}$$

we need to calculate $dLIPS_2$ and simplify $|\mathcal{T}|^2$. The former sounds more fun so I start there:

$$\begin{aligned}
dLIPS_2(k_e k_\gamma) &= (2\pi)^4\delta^4(k_\gamma + k_e - k'_\gamma - k'_e)\frac{\mathrm{d}^3k'_e}{(2\pi)^3 2E'_e}\frac{\mathrm{d}^3k'_\gamma}{(2\pi)^3 2\omega'} &(17)\\
&= \frac{1}{4(2\pi)^2 E'_e\omega'}\delta(m_e + \omega - \omega' - E'_e)\delta^3(\vec{k}_\gamma - \vec{k}'_\gamma - \vec{k}'_e)\mathrm{d}^3k'_e\mathrm{d}^3k'_\gamma &(18)\\
&= \frac{1}{16\pi^2}\frac{1}{E'_e + \omega' - \omega\cos\theta_{FT}}\omega'\mathrm{d}\Omega_{FT} &(19)\\
&= \frac{1}{16\pi^2}\frac{\omega'\mathrm{d}\Omega_{FT}}{m + \omega(1 - cos\theta_{FT})} &(20)\\
&= \frac{1}{16\pi^2}\frac{\omega'\mathrm{d}\Omega_{FT}}{m + \omega\left(\frac{m_e(\omega-\omega')}{\omega\omega'}\right)} &(21)\\
&= \frac{1}{16\pi^2}\frac{\omega'^2\mathrm{d}\Omega_{FT}}{m\omega} &(22)
\end{aligned}$$

where doing the integral over $\mathrm{d}^3k'_e$ results in the replacement $\vec{k}'_e = \vec{k}_\gamma - \vec{k}'_\gamma$. Thus, the radial integral over the energy delta function could be implemented as $\int\delta(f(x)) = \sum_{0's}|f'(x_0)|^{-1}$

$$\begin{aligned}
\left(\frac{\partial}{\partial|k'_\gamma|} = \frac{\partial}{\partial\omega'}\right)\left(E'_e + \omega' - \sqrt{s}\right) &= \frac{\partial E'_e}{\partial\omega'} + \frac{\partial\omega'}{\partial\omega'} &(23)\\
&= \frac{\omega' - \omega\cos\theta_{FT}}{E'_e} + 1 &(24)\\
&= \frac{E'_e + \omega' - \omega\cos\theta_{FT}}{E'_e} &(25)
\end{aligned}$$

We now simplify the amplitude given to us from a future calculation. For the first step of plugging in mandelstam expressions and simplification, I politely ask mathematica to handle the algebra.

$$\begin{aligned}
|\mathcal{T}|^2 &= 32\pi^2\alpha^2\left[\frac{m^2(\omega - \omega')^2}{\omega^2\omega'^2} + \frac{2m(\omega' - \omega)}{\omega\omega'} + \frac{\omega^2 + \omega'^2}{\omega\omega'}\right] &(26)\\
&= 32\pi^2\alpha^2\left[\frac{\omega^2 + \omega'^2}{\omega\omega'} - \sin^2\theta_{FT}\right] &(27)\\
&= 32\pi^2\alpha^2\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta_{FT}\right] &(28)
\end{aligned}$$

where, inspired by the $\sin^2\theta_{FT}$ in the Klein-Nishina formula, I have substituted:

$$\begin{aligned}
-\sin^2\theta_{FT} &= \cos^2\theta_{FT} - 1 &(29)\\
&= \frac{m^2(\omega' - \omega)^2}{\omega^2\omega'^2} + \frac{2m(\omega' - \omega)}{\omega\omega'} &(30)
\end{aligned}$$

Compiling all the pieces as outlined in Eq. (16),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{FT}} = \frac{1}{4m_e\omega}\frac{1}{16\pi^2}\frac{\omega'^2}{m\omega}32\pi^2\alpha^2\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta_{FT}\right] \tag{31}$$

$$= \frac{\alpha^2}{2m_e^2}\frac{\omega'^2}{\omega^2}\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta_{FT}\right] \tag{32}$$

2. Renormalization of ϕ^4

The perturbatively renormalized Lagrangian of ϕ^4 theory may be written as

$$\mathcal{L} = -\frac{1}{2}Z_\phi\left(\partial\phi\right)^2 - \frac{1}{2}Z_m m^2\phi^2 - \frac{1}{4!}\lambda Z_\lambda\tilde{\mu}^\epsilon\phi^4 \tag{33}$$

As usual, this lagrangian may be split into bare and counterterm parts:

$$\mathcal{L}_0 = -\frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \tag{34}$$

$$\mathcal{L}_{ct} = -\frac{1}{2}\left(Z_\phi-1\right)\left(\partial\phi\right)^2 - \frac{1}{2}\left(Z_m-1\right)m^2\phi^2 - \frac{1}{4!}\left(Z_\lambda-1\right)\lambda\phi^4 \tag{35}$$

$$= -\frac{1}{2}\delta Z\left(\partial\phi\right)^2 - \frac{1}{2}\delta m^2\phi^2 - \frac{1}{4!}\delta\lambda\phi^4 \tag{36}$$

a.

Here, we draw the divergent 1-loop diagrams of ϕ^4 theory. Through dimensional analysis and the structure of diagrams, it can be argued that the superficial degree of divergence of any diagram in ϕ^4 theory in $d=4$ dimensions can be summarized only by the number of external legs:

$$D = 4 - E$$

In order for a diagram to be convergent, the value of D calculated for it and all of its possible subdiagrams must be greater than 0. Here, only diagrams with two and four external legs are divergent (quadratically and logarithmically, respectively). The divergence of the tadpole diagram will be removed by a mass

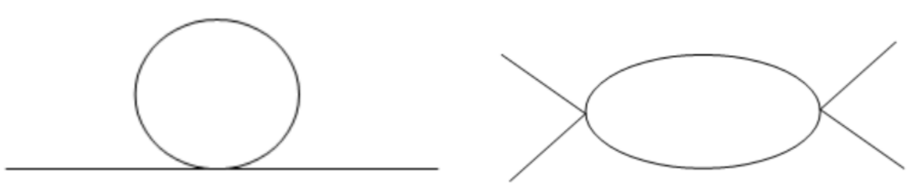


Figure 1: Divergent 1-loop diagrams in ϕ^4.

counterterm while the divergence of the fish diagram will be removed with a counterterm for the 4-pt vertex. Equivalently, this means that the former will contribute to Z_m while the latter will contribute to Z_λ.

Neither of the above 1-loop diagrams contribute to Z_ϕ because they have no dependence on external momenta. When momentum conservation prevents external momentum from flowing in a loop (such as the

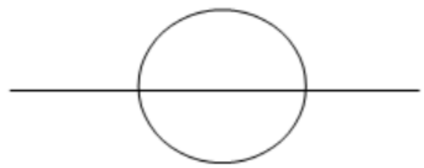


Figure 2: Divergent 2-loop diagram in ϕ^4.

case of the tadpole diagram), the diagram becomes independent of derivatives of the field and thus does not contribute to wavefunction renormalization. The fish diagram is logarithmically divergent and is thus dimensionally not able to contribute to wavefunction renormalization. In order to find a contribution to Z_ϕ, we need a divergent piece to elicit a counterterm with momentum dependence $\sim \frac{p^2}{\epsilon}$. It is reasonable that such a term may come in at two loops with three internal propagators where the diagram schematically resembles $\int \frac{\mathrm{d}^8 p}{(2\pi)^8}\frac{1}{p^6}$ and contains external momentum dependence within the loops.

b.

First, consider the evaluation of the tadpole. When dimensionally regulated with $\epsilon = 4-d$,

$$\lambda\mu^{4-d}\frac{1}{2}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2} = -\frac{i\lambda}{(4\pi)^2}\Gamma\left(-1+\frac{\epsilon}{2}\right)m^2\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \tag{37}$$

$$= \frac{i\lambda}{2(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma+1+\mathcal{O}(\epsilon)\right)m^2\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}+\right)\cdots\right) \tag{38}$$

$$= \lambda\frac{im^2}{16\pi^2}\frac{1}{\epsilon}+\frac{i\lambda m^2}{16\pi^2}\left(-\gamma+1+\ln(4\pi)+\ln\left(\frac{\mu^2}{m^2}\right)+\cdot\right) \tag{39}$$

in modified minimal subtraction $\overline{MS}$, the counterterm will remove both the divergent piece and the standard constants of $\ln(4\pi)$ and γ. This will make a new 2-pt vertex with the amplitude contribution:

$$\lambda\frac{im^2}{16\pi^2}\frac{1}{\epsilon}+\frac{i\lambda m^2}{32\pi^2}\left(-\gamma+\ln(4\pi)\right) =$$ 

to then include this new diagrammatic contribution, we introduce a mass counterterm of the form

$$\delta m^2 = \frac{\lambda m^2}{16\pi^2\epsilon}+\frac{\lambda m^2}{32\pi^2}\left(-\gamma+\ln(4\pi)\right) = (Z_m-1)m^2 \tag{40}$$

The fish diagram scales logarithmically and contributes to the renormalization of the coupling λ.

$$\lambda^2\mu^\epsilon\frac{1}{2}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2}\frac{1}{(\ell+k_1+k_2)^2+m^2} \xrightarrow{k_1+k_2=0} \lambda^2\mu^\epsilon\frac{1}{2}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+m^2)^2} \tag{41}$$

$$= \lambda^2\mu^\epsilon\frac{i}{2(4\pi)^{\frac{d}{2}}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{m^2}\right)^{2-\frac{d}{2}} \tag{42}$$

$$= \frac{i\lambda^2}{2(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right)\left(1+\frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right)+\cdots\right) \tag{43}$$

$$= \frac{i\lambda^2}{16\pi^2}\frac{1}{\epsilon}+\frac{i\lambda^2}{32\pi^2}\left(-\gamma+\ln(4\pi)\right)+\frac{i\lambda^2}{32\pi^2}\ln\left(\frac{\mu^2}{m^2}\right)+\mathcal{O}(\epsilon) \tag{44}$$

In the first line, I take only the contribution at zero external momentum as this is the location of the momentum-independent divergence (this way I get to avoid feynman parameters etc.).

$$\delta\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon} + \frac{3\lambda^2}{32\pi^2}\left(-\gamma + \ln(4\pi)\right) \tag{45}$$

where the factor of 3 comes from the degenerate calculations that would be done also for the divergences of the t and u channel diagrams.

c.

There are many ways to determine β functions. Many rely on enforcing that physical quantities (and the bare parameters expected to express them) are independent of the renormalization scale. Mathematically, this means that any shift experienced when rescaling μ must be compensated by μ dependence of the coupling constant and field strength

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right) G^{(n)} = 0 \tag{46}$$

For example, we could enforce that the 2-point Green's function or propagator be independent. Because the propagator receives no corrections to $\mathcal{O}(\lambda)$ as argued above, the first two differentials must vanish. To satisfy renormalization scale invariance, the amomalous dimension γ must also vanish.

$$\gamma(\lambda) = 0 + \mathcal{O}(\lambda^2)$$

To calculate the β function of the coupling, we may assert the same equation for the 4-point green's function. We have seen that this is a sum of the "x" diagram, the fish diagram, and the associated counterterm. With a vanishing anomalous dimension, we set the variation with respect to μ (multiplied by μ) equal to the variation with respect to λ (multiplied by the β function). The fish diagram is the only source of μ dependence. The latter simply brings down a power of 2 and we are left with

$$\beta(\lambda) = \frac{3\lambda^2}{32\pi^2}\ \mu\frac{\partial}{\partial\mu}\ln\left(\mu^2\right) \tag{47}$$

$$= \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \tag{48}$$

d.

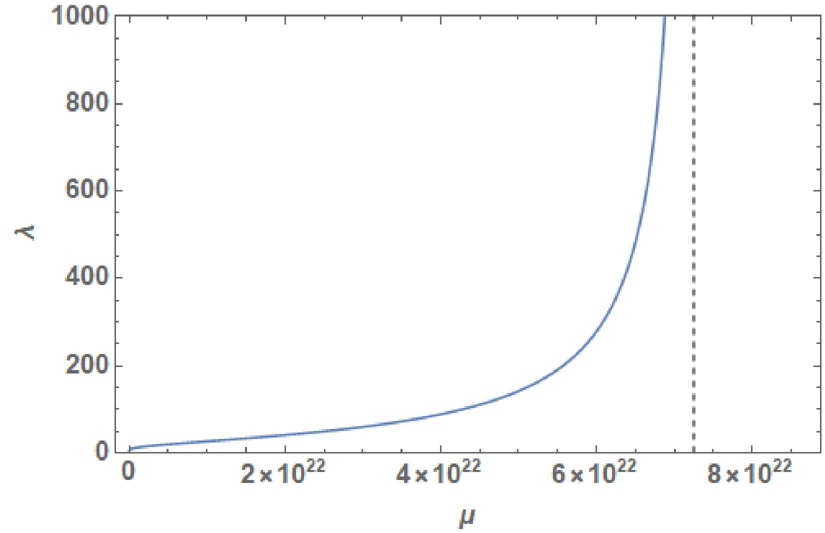
By separating variables and removing a constant through the definition of a reference scale (or plugging into mathematica, I suppose), the generic solution to the above *beta* function may be found.

$$\lambda(\mu) = \frac{1}{\frac{1}{\bar{\lambda}} - \frac{3}{16\pi^2}\ln\frac{\mu}{\bar{\mu}}} \tag{49}$$

Then specifying the reference scale to that of the problem statement,

$$\lambda(\mu) = \frac{1}{1 - \frac{3}{16\pi^2}\ln\mu} \tag{50}$$

The figure below plots the scaling of this coupling and emphasizes the presence of a "Landau pole"—a divergence of the coupling at a finite energy scale beyond which the theory cannot pass. The name comes from the famous pole in the β-function of QED. In the far IR, we see that λ slowly makes its way to to 0^+.



3. Renormalization Group

The given β function tells us how the coupling will scale with the (log of the) renormalization scale.

$$\beta(g) = \frac{\partial g}{\partial \ln \mu} = -c_1 g + c_2 g^3 \qquad c_1, c_2 > 0 \tag{51}$$

a.

Before plotting in figure 3, I choose to scale out c_1 because I expect the ratio of coefficients to be the qualitatively important quantity.

$$\beta(g) = \frac{\partial g}{\partial \ln \mu} = c_1 \left(-g + \frac{c_2}{c_1} g^3 \right) \tag{52}$$

Implications of β function sign[1]:

- For $\beta(g) > 0$, $\frac{\partial g}{\partial \ln \mu} > 0$ so that a *decrease* in μ results in a *decrease* in the coupling g.
- For $\beta(g) < 0$, $\frac{\partial g}{\partial \ln \mu} > 0$ so that a *decrease* in μ results in an *increase* in the coupling g
- For $\beta(g) = 0$, the derivative is zero and the coupling g does not scale with the renormalization scale μ.

b.

The central zero of the β function is the "Gaussian" or "trivial" fixed point where the coupling itself vanishes and we are left with a non-interacting theory. The theory explores this possibility by traveling backwards along the RG flow i.e., in the UV when beginning with a coupling inside the $\pm\frac{c_1}{c_2}$ fixed points. If traversing the UV from larger couplings, the coupling is unbounded and grows in strength.

[1] For the following statements, I assume the renormalization scale μ is a positive number so that the sign of the beta function implies equivalent properties as the sign of the derivative as $\beta(g) = \frac{\partial g}{\partial \ln \mu} = \mu \frac{\partial g}{\partial \mu}$.

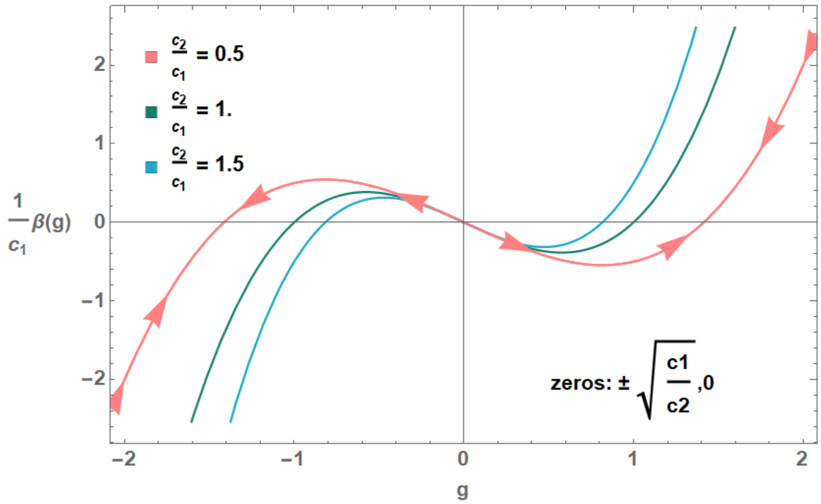


Figure 3: Scaled *beta* function shows three zeros. These zeros are fixed points of the renormalization group—locations where the coupling g stops scaling with μ. The outer two are IR-stable fixed points illustrated by the arrows showing the behaviour of g when μ is decreased in each region.

A low-energy perturbative expansion about either of the IR fixed points would result in a stable expansion, though one that is unaware of the presence of the other. Theories expanded about different fixed points can have dramatically different features: confinement, asymptotic freedom, etc. Depending on the role of this coupling in the Lagrangian, choosing one or the other of these fixed points (i.e., changing the sign of the coupling) could yield results as dramatic as creating an unbounded potential.

c.

Perturbation theory generically relies on a small expansion parameter. If we begin with a free theory near the origin and flow to low energies to attempt a perturbative expansion in g, we would find that as we lower the physical energy scale μ, the coupling grows. This is exactly what the IR arrows indicate in Figure 3. With a large coupling at low energies, we can no longer expect a perturbative treatment of the theory to be valid. However, if the IR fixed points were located at perturbatively-small values of g, there would be no problem i.e., if $\sqrt{\frac{c_1}{c_2}}$ (potentially divided by relevant mass scales in the problem depending on the units of g) is much less than 1 so that $c_2 >> c_1$.

It is also possible to think of perturbative expansions in the UV. In this case, all our arrows would flip directions and the free theory would be a stable fixed point around which a perturbative expansion could be made. This is similar to the story of perturbative QCD with asymptotic freedom: the vanishing of the coupling in the UV. Generically, reasonable perturbative expansions must be done about RG-stable fixed points.