QFT 570 Homework 3

Srednicki 10.2

The Feynman rules must instruct the construction of $i\mathcal{T}$ by summarizing the results of the LSZ.

When we did problem 3.5, we saw that a complex scalar field could be decomposed into two independent types of particles creatively named: a and b with charges +1 and -1, respectively. When we worked out the LSZ for this theory in problem 5.1, we found that φ creates a b particle or annihilates an a particles while $\varphi^\dagger$ does the opposite: creates an a and annihilates a b. While we have not done problem 9.3, it simply instructs us to consider two source types for the complex scalar field: J for the insertion of particles (or the annihilation of antiparticles) with an arrow pointing away from the source and $J^\dagger$ for the annihilation of particles (or the creation of antiparticles) with an arrow pointing towards the source. Note that these are charge arrows, not momentum arrows.

The Lagrangian of problem 9.3:

$$\mathcal{L} = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi - \frac{1}{4}Z_\lambda\lambda(\varphi^\dagger\varphi)^2 - (Z_\varphi - 1)\partial^\mu\varphi^\dagger\partial_\mu\varphi - (Z_m - 1)m^2\varphi^\dagger\varphi \tag{1}$$

looks to consist now of a four-point vertex (coupling constant λ) along with the mass and kinetic counterterms. As we have previously seen in φ^3 scalar theory, these counterterms create a new vertex where two lines meet. The difference now is that the two lines meeting in this theory will have oppositely directed charge arrows:

Modifying Srednicki's listing of the Feynman rules for scalar fields involves careful consideration of the content expressed by arrows and recalculation of the associated vertex factors.

1. Draw external lines for each incoming and outgoing particle
2. Leave one end of each external line free (for removed source), and attach the other to a vertex at which exactly FOUR lines meet (two φ and two $\varphi^\dagger$). Include extra internal lines in order to do this if necessary. Draw all possible diagrams that are topologically inequivalent
3. Alongside each incoming line, draw an arrow pointing towards the vertex. Alongside each outgoing line, draw an arrow pointing away from the vertex. These are momentum arrows and may be placed parallel but separate from the line to distinguish them from charge arrows. The momentum arrow direction may be chosen arbitrarily for internal lines as long as they are treated consistently throughout the calculation. In terms of our previous decomposition of complex scalar fields into a and b particles, these momentum arrows will be in the same direction as charge arrows for the a particles and in the opposite direction as the charge arrows for b particles. This is the key to Srednicki's suggestion that a more elegant approach reduces to a single type of arrow: basically momentum arrows that pick up a minus sign when describing b particles. The placement of these combined arrows may be summarized as

- incoming a or outgoing b: arrow towards the vertex
- incoming b or outgoing a: arrow away from the vertex

4. Assign four-momenta (b particles with a negative sign), conserving this quantity at each vertex. This constraint will fix the momenta throughout a tree diagram and leave 1 unfixed momentum for each loop present in higher-order diagrams. If using the combined arrows, this step requires enforcing that every vertex has two incoming arrows and two outgoing arrows.

5. Features in the diagram come with the following factors:
   - external line: 1
   - each vertex: $-i\lambda Z_\lambda$
   - internal line with momentum k: $\frac{-i}{k^2+m^2-i\epsilon}$

6. A diagram with L closed loops will have L internal momenta that are not fixed by the above. Integrate over each of these momenta ℓ_i with measure $\frac{\mathrm{d}^4\ell_i}{(2\pi)^4}$.

7. A loop diagram may have some left-over symmetry factors if there are exchanges of internal propagators and vertices that leave the diagram unchanged; in this case, difide the value of the diagram by the symmetry factor associated with exchanges of internal propagators and vertices.

8. Include diagrams with the counterterm vertex that connects two propagators with the same four-momentum k. As observed above, sprinkling these vertices into a diagram of a complex scalar theory changes the propagation from that of an a particle to that of a b particle. The vertex factor remains $-i\left((Z_\varphi - 1)k^2 + (Z_m - 1)m^2\right)$ which remains a process of order λ^2 as $Z_i = 1 + \mathcal{O}\left(\lambda^2\right)$.

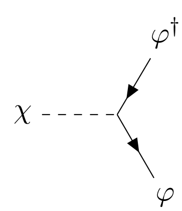
9. The values of $i\mathcal{T}$ is given by a sum over the values of all these diagrams.

Srednicki 10.3

When a 3-point vertex is present in out complex scalar theory describing the interaction of an a, a b, and a scalar

$$\mathcal{L}_1 = g\chi\varphi^\dagger\varphi$$

the corresponding vertex contains one ingoing and one outgoing arrow on the complex scalar fields



Srednicki 11.1

a

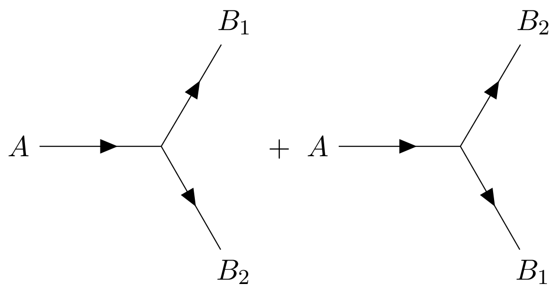
We are asked to calculate the total decay rate for an A particle to a pair of B particles governed by an interaction $\mathcal{L}_1 = gAB^2$. To begin, we start in the center of mass frame of particle A. Here, A is at rest with a 4-momentum of $(m_A, \mathbf{0})$ so that $\sqrt{s} = m_A$. Using a symmetry factor of 2 in Eq. 11.49 (for the two identical external particles), the differential of Eq. 11.48, and the value of $dLIPS$ evaluated for 2-particle final states in Eq. 11.30, we get the expression:

$$\Gamma = \frac{1}{2}\int \frac{1}{2m_A}|\mathcal{T}|^2 \frac{|\mathbf{k}_1'|}{16\pi^2\sqrt{s}}\mathrm{d}\Omega_{CM} \tag{2}$$

Note that Eq. 11.30 has indeed been evaluated in the CM frame so such combination is valid. From here, we may use the kinematic relationship of Eq. 11.3 with identical final states,

$$|\mathbf{k}_1'| = \frac{1}{2m_A}\sqrt{m_A^4 - 4m_B^2 m_A^2} \tag{3}$$

and the feynman rules to calculate the value of $\mathrm{i}\mathcal{T}$ from the single tree-level diagram with one degree of symmetry for the identical external states.



(4)

Recall that tree diagrams are generically left with no symmetry factor once sources have been removed and endpoints have been labeled. This identical particle final state has a symmetry factor in labeling itself resulting in a vertex factor of $2ig$. This makes $|\mathcal{T}|^2 = 4g^2$. Combining all these pieces,

$$\Gamma_{ABB} = \frac{1}{2}\int \frac{1}{2m_A}4g^2\frac{1}{16\pi^2 m_A}\frac{1}{2m_A}\sqrt{m_A^4 - 4m_B^2 m_A^2}\mathrm{d}\Omega_{CM} \tag{5}$$

$$= \frac{g^2}{8\pi m_A}\sqrt{1 - 4\frac{m_B^2}{m_A^2}} \tag{6}$$

As expected by the rotationally invariant initial state, the integrand is constant over the entirety of the solid angle so that $\int \mathrm{d}\Omega_{CM} = 4\pi$.

b

For the second part of this problem, the scalar final states have been modified to complex scalar final states. Your immediate thought should be that this removes a symmetry factor by treating the complex as carrying two independent, distinguishable fields. The symmetry factor that gets removed is the one associated with the labeling of the endpoints in the calculation of $|\mathcal{T}|^2$ as well as the one from Eq. 11.49. Removing both of these terms simply changes the above calculation by a factor of $\frac{1}{2}$.

$$\Gamma_{\varphi\chi^\dagger\chi} = \frac{g^2}{16\pi m_\varphi}\sqrt{1 - 4\frac{m_\chi^2}{m_\varphi^2}} \tag{7}$$

Srednicki 11.3

In this problem, we calculate the decay rate of the muon given the result of a future calculation of the spin-symmed amplitude:

$$|\mathcal{T}|^2 = 64G_F^2(k_1 \cdot k_2')(k_1' \cdot k_3') \tag{8}$$

with the unfortunate naming convention

$$k_1 = k_\mu \qquad k'_1 = k_{\bar{\nu}_e} \qquad k'_2 = k_{\nu_\mu} \qquad k'_3 = k_{e-}$$

As all final states are distinguishable, there is no symmetry factor to worry about and the decay rate may be written as

$$\Gamma_\mu = \int \frac{1}{2m_\mu} 64G_F^2 (k_1 \cdot k'_2)(k'_1 \cdot k'_3) dLIPS_3(k_1) \tag{9}$$

a

Begin by evaluating $dLIPS_3(k_1)$ from Eq. 11.23 to break off the integral over the electron's four-momentum.

$$\begin{aligned} dLIPS_3(k_1) &= (2\pi)^4 \delta^4(k_1 - k'_1 - k'_2 - k'_3)\widetilde{\mathrm{d}k'_1}\widetilde{\mathrm{d}k'_2}\widetilde{\mathrm{d}k'_3} & (10)\\ &= (2\pi)^4 \delta^4((k_1 - k'_3) - k'_1 - k'_2)\widetilde{\mathrm{d}k'_1}\widetilde{\mathrm{d}k'_2}\widetilde{\mathrm{d}k'_3} & (11)\\ &= dLIPS_2(k_1 - k'_3)\widetilde{\mathrm{d}k'_3} & (12) \end{aligned}$$

now the k'_3 dependence may be passed through the $dLIPS_2$ integral as

$$\begin{aligned} \Gamma &= \frac{32G_F^2}{m_\mu} \int \widetilde{\mathrm{d}k'_3} \int (k_1)_\mu k_2'^\mu (k'_1)_\nu k_3'^\nu dLIPS_2(k_1 - k'_3) & (13)\\ &= \frac{32G_F^2}{m_\mu} \int \widetilde{\mathrm{d}k'_3} (k_1)_\mu (k'_3)_\nu \int k_2'^\mu (k'_1)^\nu dLIPS_2(k_1 - k'_3) & (14) \end{aligned}$$

b

If a quantity is to be lorentz invariant, it must be a linear combination of all the lorentz invariants that can be made out of available variables. Here, we are looking to decompose an object that is a 2-index tensor under lorentz transformations. The available structures with this transformation property are: $g^{\mu\nu}$ and $k^\mu k^\nu$. Other tempting choices involving k_1 or k_2 are not allowed as k_1 and k_2 are integrated out. Dimensionally, we need a dimension $mass^2$ object to multiply with $g^{\mu\nu}$. As the pertinent particles (the neutrinos) have been declared massless for this calculation, the only available structure is k^2. Thus, the most general object we can make with the appropriate lorentz transformation properties (2,0) is

$$\int k_1'^\mu k_2'^\nu dLIPS_2(k) = Ak^2 g^{\mu\nu} + Bk^\mu k^\nu \tag{15}$$

where all dimensionful quantities are explicit, leaving A and B to be numerical constants[1].

c

Simply applying the masslessness constraint to the calculation of $dLIPS_2(k)$ of Eq. 11.30 (in CM):

$$\begin{aligned} \int dLIPS_2(k) &= \frac{|\mathbf{k}'_1|}{16\pi^2\sqrt{s}} \mathrm{d}\Omega_{CM} & (16)\\ &= \int \frac{s}{2\sqrt{s}} \frac{1}{16\pi^2\sqrt{s}} \mathrm{d}\Omega_{CM} & (17) \end{aligned}$$

[1]Note that this argument will become pertinent again when working with photon propagators

$$= \int \frac{1}{32\pi^2} \mathrm{d}\Omega_{CM} \tag{18}$$

$$= \frac{1}{32\pi^2} 4\pi = \frac{1}{8\pi} \tag{19}$$

d

If we first contract Eq. 11.55 with with $g_{\mu\nu}$:

$$\int (k_1')_\mu k_2'^\mu dLIPS_2(k) = 4Ak^2 + Bk^2 = (4A+B)k^2 \tag{20}$$

where I can think of $g_{\mu\nu}g^{\mu\nu}$ as the trace of the product of metric tensors. In four spacetime dimensions, this quantity is 4. To interpret the dot product, we do a bit of kinematics:

$$k = k_1 - k_3' = k_1' + k_2' \qquad k^2 = k_1'^2 + k_2'^2 + 2k_1'k_2' \overset{m=0}{=} 2k_1'k_2'$$

Then,

$$\frac{k^2}{2} \int dLIPS_2(k) = (4A+B)k^2 \tag{21}$$

so that

$$(4A+B) = \frac{1}{16\pi} \tag{22}$$

We next contract instead with $k_\mu k_\nu$ to get a second independent equation.

$$\int (k_1' \cdot k)(k_2' \cdot k) dLIPS_2(k) = (A+B)k^4 \tag{23}$$

Now a bit more kinematics to evaluate the dot products

$$(k_1' \cdot (k_1' + k_2'))(k_2' \cdot (k_1' + k_2')) = (0 + k_1' \cdot k_2')(k_2' \cdot k_1' + 0) = (k_1'k_2')^2 = k^2$$

$$\frac{k^4}{4} \int dLIPS_2(k) = (A+B)k^4 \tag{24}$$

so that

$$(A+B) = \frac{1}{32\pi} \tag{25}$$

Combining equations (22) and (25),

$$B = \frac{1}{48\pi} \qquad A = \frac{1}{96\pi} \tag{26}$$

e

To find the differential decay rate with respect to the electron energy, we continue to calculate the total decay rate and plan to identify

$$\Gamma = \int \mathrm{d}E_e \, \frac{\mathrm{d}\Gamma}{\mathrm{d}E_e}$$

$$\Gamma = \frac{32G_F^2}{m} \int \widetilde{\mathrm{d}k_3'} (k_1)_\mu (k_3')_\nu \left[\frac{1}{96\pi} k^2 g^{\mu\nu} + \frac{1}{48\pi} k^\mu k^\mu\right] \tag{27}$$

$$= \frac{32G_F^2}{m}\int \widetilde{\mathrm{d}k_3'}\left[\frac{1}{96\pi}k^2(k_1\cdot k_3') + \frac{1}{48\pi}(k_1\cdot k)(k_3'\cdot k)\right] \tag{28}$$

$$= \frac{32G_F^2}{m}\int \widetilde{\mathrm{d}k_3'}\left[\frac{1}{96\pi}(k_1-k_3')^2(k_1\cdot k_3') + \frac{1}{48\pi}(k_1\cdot(k_1-k_3'))(k_3'\cdot(k_1-k_3'))\right] \tag{29}$$

$$= \frac{32G_F^2}{m}\int \widetilde{\mathrm{d}k_3'}\left[\frac{1}{96\pi}(k_1^2-2k_1\cdot k_3')(k_1\cdot k_3') + \frac{1}{48\pi}(k_1^2-k_1\cdot k_3')(k_3'\cdot k_1)\right] \tag{30}$$

$$= \frac{32G_F^2}{m}\int \widetilde{\mathrm{d}k_3'}(k_1\cdot k_3')\left[\left(\frac{1}{96\pi}+\frac{1}{48\pi}\right)k_1^2 - \left(\frac{2}{96\pi}+\frac{1}{48\pi}\right)k_1\cdot k_3'\right] \tag{31}$$

$$= \frac{G_F^2}{m\pi}\int \widetilde{\mathrm{d}k_3'}(k_1\cdot k_3')\left[k_1^2 - \frac{4}{3}k_1\cdot k_3'\right] \tag{32}$$

$$= \frac{G_F^2}{\pi}\int \widetilde{\mathrm{d}k_3'}E_e\left[m^2 - \frac{4}{3}E_e m\right] \tag{33}$$

$$= \frac{G_F^2}{\pi}\int \frac{d^3k_3'}{(2\pi)^3 2E_e}E_e\left[m^2 - \frac{4}{3}E_e m\right] \tag{34}$$

$$= \frac{G_F^2}{\pi^4 2^4}\int E_e^2\mathrm{d}E_e\mathrm{d}\Omega\ \left[m^2 - \frac{4}{3}E_e m\right] \tag{35}$$

$$= \frac{mG_F^2}{4\pi^3}\int \mathrm{d}E_e\ \left[E_e^2 m - \frac{4}{3}E_e^3\right] \tag{36}$$

where I have used $k_1\cdot k_3' = -E_e m$ as evaluated in the CM frame. Identifying the integrand as the differential decay rate:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_e} = \frac{mG_F^2}{4\pi^3}\left[E_e^2 m - \frac{4}{3}E_e^3\right] \tag{37}$$

A quick kinematics exercise to conserve energy in the scenario that the two neutrinos emerge in one direction (sharing equal momentum) and the electron in the opposite direction will result in a calculation of the maximal electron energy, $\frac{m_\mu}{2}$.

f

Performing the integral is again straightforward

$$\Gamma = \frac{mG_F^2}{4\pi^3}\left[\frac{1}{3}E_e^3 m - \frac{1}{3}E_e^4\right]\Big|_0^{\frac{m}{2}} \tag{38}$$

$$= \frac{mG_F^2}{4\pi^3}\left[\frac{m^4}{3\cdot 2^3} - \frac{m^4}{3\cdot 2^4}\right] \tag{39}$$

$$= \frac{m^5 G_F^2}{192\pi^3} \tag{40}$$

g

Evaluating with numbers:

$$G_F = \sqrt{\frac{192\Gamma\pi^3}{m^5}} = \sqrt{\frac{192\frac{6.582*10^{-25}GeV/s}{2.197*10^{-6}s}\pi^3}{0.10566^5}} = 1.164*10^{-5}GeV \tag{41}$$

h

The energy spectrum of the electron:

$$P(E_e) \equiv \Gamma^{-1}\frac{\mathrm{d}\Gamma}{\mathrm{d}E_e} \tag{42}$$

$$= \frac{192\pi^3}{m^5 G_F^2}\frac{mG_F^2}{4\pi^3}\left[E_e^2 m - \frac{4}{3}E_e^3\right] \tag{43}$$

$$= \frac{48}{m^4}\left[E_e^2 m - \frac{4}{3}E_e^3\right] \tag{44}$$

$$= \frac{48}{m}\left[\frac{E_e}{m}^2 - \frac{4}{3}\frac{E_e}{m}^3\right] \tag{45}$$

To change variables in a probability distribution we will need a jacobian to assure the proper scaling of area elements:

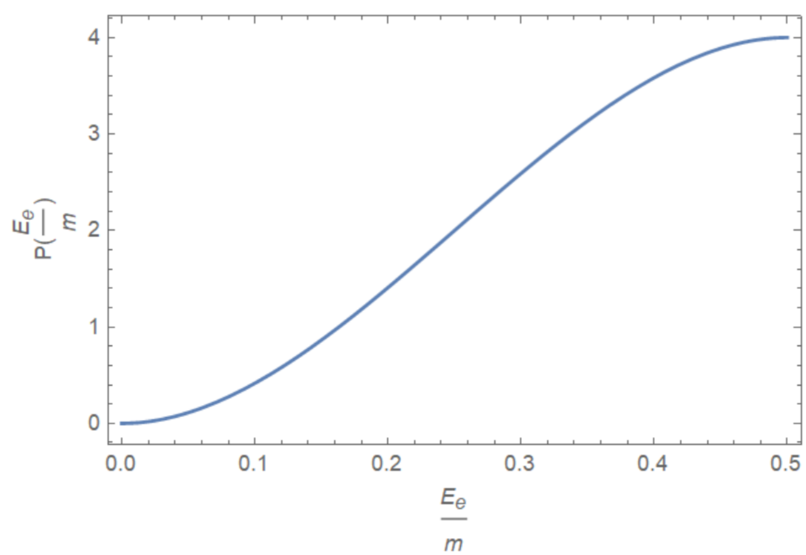
$$p(X) = p(Y) \times |\frac{\mathrm{d}Y}{\mathrm{d}X}|$$

The jacobian here will be

$$|\mathrm{d}E_e\frac{\mathrm{d}\frac{E_e}{m}}{=}m$$

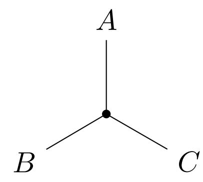
thus,

$$P\left(\frac{E_e}{m}\right) = 48\left[\frac{E_e}{m}^2 - \frac{4}{3}\frac{E_e}{m}^3\right] \tag{46}$$



Srednicki 11.4

For this theory of three scalar fields, we see that all three are propagating, massive fields with a single 3-point interaction involving one of each carrying a vertex factor of ig. the only vertices that should appear in diagrams for perturbative expansions of e.g., correlation functions within this theory are:

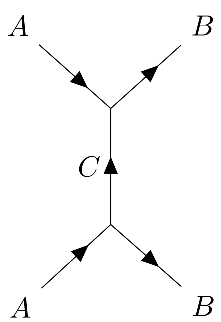


Considering only tree-level diagrams severely limits the non-vanishing amplitudes. For an initial state with two identical particles (AA), the s-channel cannot be made as an AA? vertex is not a valid interaction. The t- and u-channel diagrams are possible and connect the two vertices. This forces the external particles to also be the same type such that $AA \to CC$ or $AA \to BB$ will have a non-zero tree-level scattering amplitude but

$$\mathcal{T}^{tree}_{AA\to AA} = 0$$
$$\mathcal{T}^{tree}_{AA\to AB} = 0$$
$$\mathcal{T}^{tree}_{AA\to BC} = 0$$

and the diagrams contributing to $AA \to BB$ are

$\mathcal{T}^{tree}_{AA\to BB} =$



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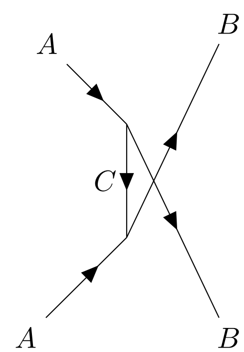
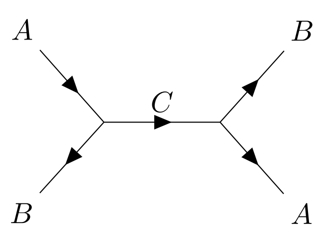


$$\mathcal{T}^{tree}_{AA\to BB} = g^2\left(\frac{1}{m_C^2 - t} + \frac{1}{m_C^2 - u}\right) \tag{47}$$

Remaining to be considered are the processes containing an initial state of dissimilar particles. Now, the s-channel diagram is possible and it is obvious that the particles in the final state must be the same two types as those in the initial state.



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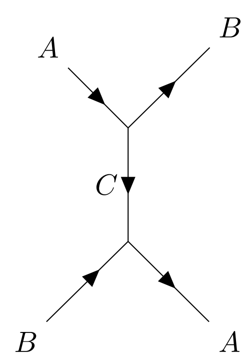

(48)

$$\mathcal{T}^{tree}_{AB\to AB} = g^2\left(\frac{1}{m_C^2 - s} + \frac{1}{m_C^2 - t}\right) \tag{49}$$

Finally, it should now be clear that

$$\mathcal{T}^{tree}_{AB\to AC} = 0$$

Srednicki eq. 14.27

As usual for these types of evaluations, we will switch to hyper-spherical coordinates. The main content of the integral is then shifted to the radial integral while the angular integral may be quickly evaluated for arbitrary dimensions in as (see Peskin and Schroeder pg. 249 for more details)

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{50}$$

I will now split the radial integral and do a couple changes of variables with the goal of identifying the beta function:

$$\beta(a,b) = \int_0^1 \mathrm{d}t\ t^{a-1}(1-t)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\begin{aligned}
\int \frac{\mathrm{d}^d\bar{q}}{(2\pi)^d}\frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} &= \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}\int \mathrm{d}\bar{q}\ \frac{(\bar{q}^2)^{a+\frac{d}{2}-1}}{(\bar{q}^2+D)^b} && (51)\\
&= \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\frac{1}{2}\int \mathrm{d}(\bar{q}^2)\ \frac{(\bar{q}^2)^{a+\frac{d}{2}-1}}{(\bar{q}^2+D)^b} && (52)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_0^1 \frac{\mathrm{d}x}{D}\ \frac{(\frac{D}{x}-D)^{a+\frac{d}{2}-1}}{(\frac{D}{x})^{b-2}} && (53)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_0^1 \frac{\mathrm{d}x}{D}\ \frac{D^{a+\frac{d}{2}-1}x^{-a-\frac{d}{2}+1}(1-x)^{a+\frac{d}{2}-1}}{(\frac{D}{x})^{b-2}} && (54)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(\frac{1}{D}\right)^{1-a-\frac{d}{2}+1+b-2}\int_0^1 \mathrm{d}x\ x^{b-a-2+1-\frac{d}{2}}(1-x)^{a+\frac{d}{2}-1} && (55)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(\frac{1}{D}\right)^{b-a-\frac{d}{2}}\int_0^1 \mathrm{d}x\ x^{b-a-\frac{d}{2}-1}(1-x)^{a+\frac{d}{2}-1} && (56)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(\frac{1}{D}\right)^{b-a-\frac{d}{2}}\beta\left(b-a-\frac{d}{2}, a+\frac{d}{2}\right) && (57)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(\frac{1}{D}\right)^{b-a-\frac{d}{2}}\frac{\Gamma\left(b-a-\frac{d}{2}\right)\Gamma\left(a+\frac{d}{2}\right)}{\Gamma\left(b\right)} && (58)
\end{aligned}$$

where in the second line I have changed variables in the radial integral to $\bar{q}^2$ and in the third line I have changed variables to $x = \frac{D}{\bar{q}^2+D}$.

Srednicki 14.3

This problem is very similar to part b of 11.3 above. The goal is to identify lorentz tensors with the same transformation properties as the quantity of interest and inform a linear combination of them through contraction.

a

The first equation, regardless of its lorentz structure, must vanish as it is an odd function of q.

The second equation is a 2-index lorentz tensor and thus must have a $g^{\mu\nu}$ tin its decomposition. Earlier, we also included a term $\sim k^\mu k^\nu$ but because the q momenta is being integrated, there is no remaining momentum for the creation of this term. Thus we have a linear combination of only one term

$$\int \mathrm{d}^d q \; q^\mu q^\nu f(q^2) = \bar{C}_2 g^{\mu\nu} \tag{59}$$

To extract the coefficient, we contract with $g^{\mu\nu}$

$$\int \mathrm{d}^d q \; q^2 f(q^2) = \bar{C}_2(d) \tag{60}$$

where I used the fact that the self-contracted metric tensor, being the trace of its matrix product, is equal to the spacetime dimension, d. Now identifying the value of C_2 as it appears in Eq. 14.53,

$$C_2 = \frac{1}{4} \tag{61}$$

b

Now, there are 3 ways to contract the pair of 2 metric tensors needed to fully contract the 4-index integrand. One can consider also the four-index levi-cevita, but since the LHS is symmetric under interchange of indices, the contribution will naturally vanish.

$$\int \mathrm{d}^d q \; q^\mu q^\nu q^\rho q^\sigma f(q^2) = A g^{\mu\nu} g^{\rho\sigma} + B g^{\mu\rho} g^{\nu\sigma} + C g^{\mu\sigma} g^{\nu\rho} \tag{62}$$

Contracting with the expression $g_{\mu\nu} g_{\rho\sigma}$

$$g_{\mu\nu} g_{\rho\sigma} g^{\mu\nu} g^{\rho\sigma} = d^2 \qquad g_{\mu\nu} g_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = g_{\nu\mu} g^{\mu\rho} g_{\rho\sigma} g^{\sigma\nu} = d$$

where the latter can be thought of as

$$\mathrm{Tr}\left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}\right] = d$$

Now,

$$\int \mathrm{d}^d q \; q^4 f(q^2) = Ad^2 + Bd + Cd = Ad + Bd^2 + Cd = Ad + Bd + Cd^2 \tag{63}$$

implying that all three coefficients, $\{A, B, C\}$, must be equal

$$\int \mathrm{d}^d q \; q^4 f(q^2) = X(d^2 + 2d) \tag{64}$$

so that the original expression:

$$\int \mathrm{d}^d q \; q^\mu q^\nu q^\rho q^\sigma f(q^2) = \frac{\int \mathrm{d}^d q \; q^4 f(q^2)}{d^2 + 2d} \left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right) \tag{65}$$