Physics 570 – Autumn 2017

Solution set #2

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1. (a) Under rotations we have $\mathbf{k}' = R\mathbf{k}$, where $R \in$SO(3) is a rotation matrix. Therefore

$$d^3k' = \left|\frac{\partial k'_i}{\partial k_j}\right| d^3k = |R_{ij}|\, d^3k = d^3k \tag{1}$$

since $\det R = 1$. Therefore since ω is invariant under rotation, $d\tilde{k}$ is.

(b) Since $d\tilde{k}$ is invariant under rotations, we only need to show now that it is invariant under boosts in the z direction to show invariance under any Lorentz transformation. Under a z boost we have

$$k^1 \to k^1\ , \quad k^2 \to k^2\ , \quad k^3 \to Ck^3 + S\omega\ , \omega \to C\omega + Sk^3\ , \tag{2}$$

where

$$C \equiv \cosh\theta\ , \quad S \equiv \sinh\theta\ , \quad \omega \equiv \sqrt{\mathbf{k}\cdot\mathbf{k} + m^2}\ , \tag{3}$$

thus

$$d\tilde{k} \to \frac{d^3k}{(2\pi)^3 2(C\omega + Sk^3)} \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ S\frac{k^1}{\omega} & S\frac{k^2}{\omega} & C + S\frac{k^3}{\omega}\end{bmatrix} = \frac{d^3k\left(C + S\frac{k^3}{\omega}\right)}{(2\pi)^3 2(C\omega + Sk^3)} = \frac{d^3k}{(2\pi)^3 2\omega} = d\tilde{k}\ , \tag{4}$$

proving Lorentz invariance of $d\tilde{k}$.

2.

$$\begin{aligned}\left(-\partial_\mu\partial^\mu + m^2\right) T\left(\phi(x)\phi(y)\right) &= \partial_t^2 T\left(\phi(x)\phi(y)\right) - T\left((\nabla^2 + m^2)\phi(x)\phi(y)\right) \\ &= \partial_t\left(T\left(\dot\phi(x)\phi(y)\right) + \underbrace{\delta(x^0 - y^0)[\phi(x), \phi(y)]}_{=0}\right) - T\left((\nabla^2 + m^2)\phi(x)\phi(y)\right) \\ &= \delta(x^0 - y^0)\underbrace{[\dot\phi(x), \phi(y)]}_{=-i\delta^3(\mathbf{x}-\mathbf{y})} - \underbrace{T\left((\partial_\mu\partial^\mu + m^2)\phi(x)\phi(y)\right)}_{=0} = -i\delta^4(x - y)\end{aligned} \tag{5}$$

So $c = -i$.

3. The generic Green's function can be written as

$$\Delta(x - y) = \lim_{\eta_\pm}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{-(k^0 - \omega_{\mathbf{k}} - i\eta_+)(k^0 + \omega_{\mathbf{k}} - i\eta_-)} \tag{6}$$

where we have yet to specify the limits on $\eta_\pm$. Since $ikx = -ik^0(x^0 - y^0) + i\mathbf{k}\cdot(\mathbf{x} - \mathbf{y})$, we close the contour in the upper half plane when $(x^0 - y^0) < 0$ and in the lower half plane when $(x^0 - y^0) > 0$. For the retarded Green's function we therefore require no poles in the lower half plane, or $\eta_\pm \to 0^+$, while for the advanced Green's function we want both poles in the lower half plane and $\eta_\pm \to 0^-$. The Feynman propagator in contrast has $\eta_\pm \to 0^\mp$.

4. We can write

$$\frac{1}{x-i\epsilon} = \frac{x+i\epsilon}{x^2+\epsilon^2} = \frac{x}{x^2+\epsilon^2} + i\frac{\epsilon}{x^2+\epsilon^2} \ . \tag{7}$$

It is tempting to assume the second term goes to zero as $\epsilon \to 0^+$ but that is not true at $x = 0$. Note that

$$\int_{-\infty}^{\infty} \frac{\epsilon}{x^2+\epsilon^2}\, dx = \tan^{-1} x \Big|_{-\infty}^{\infty} = \pi \ , \tag{8}$$

so

$$\Im \frac{x+i\epsilon}{x^2+\epsilon^2} = \pi\delta(x) \ . \tag{9}$$

As for the real part, it is tempting to write it as $1/x$. However, note that if $F(x)$ is a smooth function in the neighborhood of $x = 0$, then the integral

$$\int_{-\infty}^{\infty} F(x)\Re\left[\frac{1}{x+i\epsilon}\right] dx = \lim_{\epsilon\to 0^+} \left[\left(\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{\infty}\right) F(x) \left[\frac{x}{x^2+\epsilon^2}\right] dx\right] \tag{10}$$

Assuming $F(x) = f_0 + f_1 x + \frac{1}{2} f_2 x^2 + \ldots$ then in the middle integral, then the leading nonzero contribution is

$$\int_{-\epsilon}^{\epsilon} F(x) = \int_{-\epsilon}^{\epsilon} f_1 x \frac{x}{x^2+\epsilon^2} + O(\epsilon^2) = f_1\epsilon(2-\pi/2) + O(\epsilon^2) \xrightarrow{\epsilon\to 0} 0 \ . \tag{11}$$

Therefore

$$\int_{-\infty}^{\infty} F(x)\Re\left[\frac{1}{x+i\epsilon}\right] dx = \fint_{-\infty}^{\infty} \frac{F(x)}{x}\, dx = \lim_{\epsilon\to 0^+} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \frac{F(x)}{x}\, dx \ , \tag{12}$$

where $\fint$ denotes the Cauchy Principal Value convention for integrating over a $1/x$ singularity.

5. A number of ways to do this. One way: We know that $a_{\mathbf{k}}|0\rangle = 0 \ \forall k$. From Srednicki eq 3.20 we have

$$\tilde{\phi}(\mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}, 0) = \frac{1}{2\omega_{\mathbf{k}}}\left(a_{\mathbf{k}} + a^\dagger_{-k}\right) \ , \qquad \tilde{\Pi}(\mathbf{k}) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \Pi(\mathbf{x}, 0) = \frac{i}{2}\left(-a_{\mathbf{k}} + a^\dagger_{-k}\right) \ , \tag{13}$$

so that

$$a_{\mathbf{k}} = \omega_{\mathbf{k}}\tilde{\phi}(\mathbf{k}) + i\tilde{\Pi}(\mathbf{k}) \ . \tag{14}$$

The standard commutation relation

$$[\phi(\mathbf{x}, 0), \Pi(\mathbf{y}, 0)] = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{15}$$

implies from eq. (13) that

$$\left[\tilde{\phi}(\mathbf{k}), \tilde{\Pi}(\mathbf{q})\right] = i(2\pi)^3\delta^3(\mathbf{k} + \mathbf{q}) \ , \tag{16}$$

and therefore we can represent $\tilde{\Pi}(\mathbf{q})$ as the functional derivative

$$\tilde{\Pi}(\mathbf{q}) = -i\frac{\delta}{\delta\tilde{\phi}(-\mathbf{q})} \ . \tag{17}$$

Then our condition $a_{\mathbf{k}}|0\rangle = 0\ \forall k$ leads to a first order functional differential equation for the ground state wave function

$$\left(\frac{\delta}{\delta\tilde{\phi}(-\mathbf{k})} + \omega_{\mathbf{k}}\tilde{\phi}(\mathbf{k})\right)\Psi[\tilde{\phi}] = 0 \quad \Longrightarrow \quad \frac{1}{\Psi[\tilde{\phi}]}\frac{\delta\Psi[\tilde{\phi}]}{\delta\tilde{\phi}(-\mathbf{k})} = -\omega_{\mathbf{k}}\tilde{\phi}(\mathbf{k}) \tag{18}$$

with solution

$$\Psi[\tilde{\phi}] = e^{-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega_{\mathbf{k}}\tilde{\phi}(\mathbf{k})\tilde{\phi}(-\mathbf{k})}\ , \tag{19}$$

where I am using $\tilde{\phi}(\mathbf{k})$ to here to denote the eigenvalue of the operator $\tilde{\phi}(\mathbf{k})$. This is analogous to what we do in quantum mechanics, where we use "x" interchangeably for the operator and its eigenvalue; Srednicki calls the eigenvalue $\tilde{\phi}(k) \to \tilde{A}(\mathbf{k})$.

6. Srednicki 9.1: The answers are already given in the book. If you do not understand how they were obtained, come talk to David or Natalie.