Physics 570 – Autumn 2017

Problem set #6 **Due Friday, 12/8/2017**

---

1. In lecture we saw that irreducible representations of the Lorentz group transform as the (j_A, j_B) representation under $SU(2)_A \times SU(2)_B$, and are $(2j_A+1)(2j_B+1)$ dimensional. A field in the (j_A, j_B) representation transforms as

$$\phi_{is} \to \left[e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{A}}\right]_{ij} \left[e^{i(\vec{\theta}-i\vec{\omega})\cdot\vec{B}}\right]_{st} \phi_{jt} \equiv D^{j_A}_{ij}(\vec{\theta},\vec{\omega}) D^{j_B}_{st}(\vec{\theta},\vec{\omega})\, \phi_{jt} \ , \tag{1}$$

where $\vec{A}$ are the three $(2j_A+1)\times(2j_A+1)$ generators of $SU(2)$ in the j_A representation, and $\vec{B}$ are the three $(2j_B+1)\times(2j_B+1)$ generators of $SU(2)$ in the j_B representation. In this problem, consider a field ϕ transforming as the 4-dimensional $(\frac{1}{2}, \frac{1}{2})$ representation; in this case $\vec{A} = \vec{B} = \vec{\sigma}/2$, where $\sigma_{1,2,3}$ are the Pauli matrices. You can refer to the notes I posted on the Lorentz group.

(a) Since ϕ has two indices, both running over two values, you can consider it to be a 2×2 matrix. Define $\phi = \chi\sigma_2$. Show that in matrix notation, χ transforms as

$$\chi \to D^{j_A}(\vec{\theta},\vec{\omega})\chi \left(D^{j_B}(\vec{\theta},\vec{\omega})\right)^{-1} \ . \tag{2}$$

(Hint: show that a property of Pauli matrices is $\sigma_2\sigma_i^T\sigma_2 = -\sigma_i$, where the "^{T}" means transpose.)

(b) Show that $\det\chi$ is invariant under Lorentz transformations.

(c) Write

$$\chi = p_\mu \sigma^\mu \ , \qquad \sigma^\mu = \{1, \vec{\sigma}\} \tag{3}$$

where "1" means the 2×2 unit matrix and p_μ are four real numbers. Compute $\det\chi$ in terms of the p_μ. I have used the suggestive notation that p_μ is a 4-vector; does the invariance of $\det\chi$ make sense when expressed in terms of p_μ?

(d) Show that under rotations ($\vec{\omega}=0$), p_μ transforms as a 4-vector under rotations, namely

$$\chi \to D^{j_A}(\vec{\theta},0)\chi\left(D^{j_B}(\vec{\theta},0)\right)^{-1} \qquad \Longrightarrow \qquad p_\mu \to \left[e^{i\vec{\theta}\cdot\vec{J}}\right]_\mu^{\ \nu} p_\nu \tag{4}$$

where the $\vec{J}$ matrices in the 4-vector representation are given in eq. (9) of my notes. For this problem you may either show it in general analytically, or using Mathematica, or you can show it analytically for just a rotation about the z-axis.

(e) Repeat the above calculation for pure boosts ($\vec{\theta}=0$), showing that

$$\chi \to D^{j_A}(0,\vec{\omega})\chi\left(D^{j_B}(0,\vec{\omega})\right)^{-1} \qquad \Longrightarrow \qquad p_\mu \to \left[e^{i\vec{\omega}\cdot\vec{K}}\right]_\mu^{\ \nu} p_\nu \tag{5}$$

where the $\vec{K}$ matrices in the 4-vector representation are related to those given in eq. (9) of my notes by a minus sign, due the raised and lowered indices being switched here. For this problem you may again either show it in general analytically, or using Mathematica, or you can show it analytically for just a boost along the z-axis.