Physics 570 – Autumn 2017

Problem set #5 **Due Monday, 11/27/2017**

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1. Consider the model in Srednicki 28.3 in $d = 6$ dimensions.

   (a) Draw all the tree level diagrams contributing to $\chi\chi \to \chi\chi$ scattering and compute the scattering amplitude in the center of mass frame. Use $k_{1,2}$ for the incoming momenta and $k'_{1,2}$ for the outgoing momenta.

   (b) Expand your above result to second order in the χ momenta.

   (c) Assume that the the mass of the ϕ is much greater than the mass of the χ: $m \gg M$. At the scale $\mu = m$ we will integrate the ϕ field out of the theory and match onto an effective theory involving only the χ field. We will treat the effective theory in an expansion in M^2/m^2 and k^2/m^2, where k^2 any Lorentz invariant term second order in the χ momenta. Consider the expansion for χ^4 operators to second order in the expansion:

   $$\mathcal{L}_{\rm EFT} = -\tfrac{1}{2}\left(\partial_\mu\chi\partial^\mu\chi + M^2\chi^2\right) - \tfrac{a}{m^2}\tfrac{\chi^4}{4!} - \tfrac{bM^2}{m^4}\tfrac{\chi^4}{4!} - \tfrac{c}{m^4}\tfrac{(\partial_\mu\chi\partial^\mu\chi)\chi^2}{4} + \dots \qquad (1)$$

   where the ellipsis refers to operators with more derivatives or higher powers of χ, and a, b, c are pure numbers independent of m and M. Since we are only looking at tree level matching, $d = 6$ and the Z factors are all set to one. Compute the tree level $\chi\chi \to \chi\chi$ scattering in this effective theory in the center of mass frame up to order k^2/m^2.

   (d) Compare your results from the two theories and find the matching conditions for a, b, c.

   (e) Why didn't we need to include a fourth operator of the form $(\partial_\mu\partial^\mu\chi)\chi^3$?

2. In this problem, repeated indices are summed. Consider the 2-dimensional simple harmonic oscillator with Hamiltonian

   $$H = \tfrac{1}{2}\left(p_ip_i + \omega^2 x_ix_i\right) = \omega\left(a_i^\dagger a_i + 1\right) \qquad (2)$$

   where the indices i, j run over $1, 2$, I have set $\hbar = m = 1$, and the ladder operators have the usual definition

   $$a_i = \frac{(x_i + ip_i)}{\sqrt{2}}\ , \qquad a_i^\dagger = \frac{(x_i - ip_i)}{\sqrt{2}}\ , \qquad (3)$$

   and they satisfy

   $$[a_i, a_j] = [a_i^\dagger a_j^\dagger] = 0\ , \qquad [a_i, a_j^\dagger] = \delta_{ij}\ . \qquad (4)$$

   We know that the number operators $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$ take on non-negative integer values, and that the eigenstates of the H are the states $|n_1, n_2\rangle$ where

   $$N_1|n_1, n_2\rangle = n_1|n_1, n_2\rangle\ , \quad N_2|n_1, n_2\rangle = n_2|n_1, n_2\rangle\ , \quad H|n_1, n_2\rangle = \hbar\omega(n_1 + n_2 + 1)|n_1, n_2\rangle\ , \qquad (5)$$

   with $n_{1,2} = 0, 1, 2, \dots$. Thus the n^{th} energy level has energy $E_n = \hbar\omega(n + 1)$ with $n = (n_1 + n_2) = 0, 1, 2 \dots$.

(a) Give a formula for degeneracy of the n^{th} energy level of the system in terms of n.

(b) Show that if I define the operators $O_A = a^\dagger A a$, $O_B = a^\dagger B a$, where A and B are 2×2 matrices, then their commutator is given by

$$[O_A, O_B] = a^\dagger [A, B] a \ . \tag{6}$$

(c) Consider the operators

$$Q_\alpha = \tfrac{1}{2} a_i^\dagger \sigma_{ij}^\alpha a_j \ , \qquad \alpha = 1, 2, 3 \ , \tag{7}$$

where σ^α are the Pauli matrices. Show that these operators commute with the Hamiltonian and obey the SU(2) algebra familiar to you from studying angular momentum in quantum mechanics, namely

$$[Q_\alpha, H] = 0 \ , \qquad [Q_\alpha, Q_\beta] = i\epsilon_{\alpha\beta\gamma} Q_\gamma \ . \tag{8}$$

(d) Show that $Q^2 \equiv Q_\alpha Q_\alpha$ and Q_3 satisfy

$$[Q^2, H] = [Q_3, H] = [Q^2, Q_3] = 0 \ , \tag{9}$$

and that therefore eigenstates of the energy can be simultaneous eigenstates of Q^2 and Q_3.

(e) Express the Q^2 and Q_3 operators in terms of $N_{1,2}$. For Q^2, use the algebraic relation for Pauli matrices

$$\sigma_{ij}^\alpha \sigma_{k\ell}^\alpha = 2 \left(\delta_{i\ell} \delta_{jk} - \tfrac{1}{2} \delta_{ij} \delta_{k\ell} \right) \ . \tag{10}$$

(f) Use the above results to show that the energy eigenstates with energy E_n may be written as $|j, m\rangle$ where

$$Q^2 |j, m\rangle = j(j+1) |j, m\rangle \ , \qquad Q_3 |j, m\rangle = m |j, m\rangle \ . \tag{11}$$

What is E_n in terms of j and m? What possible values can j and m take? Is this consistent with your answer to part (a)?

(g) The H also commutes with the angular momentum operator $L_z = (x_1 p_2 - x_2 p_1)$. Show that L_z can be rewritten as $a_i^\dagger X_{ij} a_j$ for some matrix X and relate L_z to the Q_α. Are the $|n_1, n_2\rangle$ eigenstates of L_z? Are the $|j, m\rangle$ eigenstates of L_z? What are all of the eigenvalues of L_z in the n^{th} energy level?

3. Srednicki 22.1