Lorentz group for Physics 570 (Fall 2017)

Consider Lorentz transformations, for which the defining representation is 4-dimensional. This is the group of real matrices Λ which satisfy

$$\Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}\eta_{\alpha\beta} = \eta_{\rho\sigma} \ , \qquad \eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}_{\alpha\beta} \ , \qquad \det\Lambda = 1 \ . \tag{1}$$

The metric is used to raise and lower indices, from which it follows that $\eta^{\alpha}_{\ \beta}$ and $\eta_{\alpha}^{\ \beta}$ are the unit matrix. With this definition, the inner product between two 4-vectors, $v^{\alpha}\eta_{\alpha\beta}w^{\beta}$, is preserved under the Lorentz transformations $v \to \Lambda v$ and $w \to \Lambda w$, where Λ is a real, 4×4 matrix.

We can write

$$\Lambda = e^{i\theta_{\mu\nu}X^{\mu\nu}} \ , \qquad X^{\mu\nu} = -X^{\nu\mu} = -i\frac{\partial\Lambda}{\partial\theta_{\mu\nu}}\Big|_{\theta=0} \tag{2}$$

where the $\theta_{\mu\nu} = -\theta_{\nu\mu}$ is an antisymmetric 4×4 matrix containing the six real parameters for 3 rotations and 3 boosts. Since Λ is a real transformation matrix, the six independent $X^{\mu\nu}$ are all imaginary 4×4 matrices. Expanding eq. (1) to linear order in the $\theta_{\mu\nu}$, one finds the $X^{\mu\nu}$ must satisfy

$$0 = i\theta_{\mu\nu}\left[(X^{\mu\nu})^{\alpha}{}_{\rho}\eta^{\beta}{}_{\sigma} + \eta^{\alpha}{}_{\rho}(X_a)^{\beta}{}_{\sigma}\right]\eta_{\alpha\beta} = i\theta_{\mu\nu}\left[(X^{\mu\nu})_{\sigma\rho} + (X^{\mu\nu})_{\rho\sigma}\right] \ , \tag{3}$$

or:

$$(X^{\mu\nu})_{\rho\sigma} + (X^{\mu\nu})_{\sigma\rho} = 0. \tag{4}$$

In other words, *with both indices lowered*, the $X^{\mu\nu}$ matrices are antisymmetric.

A simple basis for the 4×4 matrices satisfying eq. (4) (imaginary and antisymmetric in both $\{\mu,\nu\}$ and $\{\alpha,\beta\}$) is

$$(X^{\mu\nu})_{\alpha\beta} = -i\left(\eta^{\mu}_{\alpha}\eta^{\nu}_{\beta} - \eta^{\mu}_{\beta}\eta^{\nu}_{\alpha}\right) \tag{5}$$

so that the matrices we want with an upper and lower index are

$$(X^{\mu\nu})^{\alpha}{}_{\beta} = -i\left(\eta^{\mu\alpha}\eta^{\nu}{}_{\beta} - \eta^{\mu}{}_{\beta}\eta^{\nu\alpha}\right) \tag{6}$$

This provides our defining representation; we can now compute the commutation relations for these matrices in eq. (6) and arrive at the abstract algebra for the Lorentz group:

$$[X^{\mu\nu},\, X^{\rho\sigma}] = i\left(\eta^{\mu\rho}X^{\nu\sigma} - \eta^{\nu\rho}X^{\mu\sigma} - \eta^{\mu\sigma}X^{\nu\rho} + \eta^{\nu\sigma}X^{\mu\rho}\right) \ . \tag{7}$$

It is convenient to identify the three rotations and three boosts, so we define

$$J_i = \frac{1}{2}\epsilon_{ijk}X^{jk} \ , \qquad K_i = X^{i0} \ , \qquad \Lambda = e^{i\vec{\theta}\cdot\vec{J} + i\vec{\omega}\cdot\vec{K}} \tag{8}$$

analogous to how the B_i and E_i fields are defined in terms of $F^{\mu\nu}$ in electromagnetism. In the representation eq. (5) this yields

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} , \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} , \tag{9}$$

en exponentiated we get the familiar rotation and boost matrices, such as

$$e^{i\theta J_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad e^{i\omega K_3} = \begin{pmatrix} \cosh\omega & 0 & 0 & \sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix} , \tag{10}$$

where θ is the rotation angle, and ω is the boost rapidity, with $\gamma = \cosh\omega$, $\beta = \tanh\omega$.

With these definitions of J_i and K_i our abstract algebra eq. (7) implies the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ , \quad [J_i, K_j] = i\epsilon_{ijk} K_k \ , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k \ . \tag{11}$$

If we can find a d-dimensional matrix representation of this algebra (call the representation "R"), then a Lorentz transformation will be given by the d-dimensional matrix

$$D_R(\vec{\theta}, \vec{\omega})^\mu{}_\nu = \left[e^{i\left(\vec{\theta}\cdot\vec{J} + \vec{\omega}\cdot\vec{K}\right)} \right]^\mu{}_\nu \ . \tag{12}$$

The goal is to find all the finite dimensional representations.

This algebra eq. (11) should look reminiscent of $SU(2)$. Next we define six generators $\vec{A}$ and $\vec{B}$ with less physical meaning but a simpler algebra[1]:

$$A_i \equiv \frac{1}{2}(J_i - iK_i) \qquad B_i \equiv \frac{1}{2}(J_i + iK_i) \tag{13}$$

From eq. (11) it follows that $\vec{A}$ and $\vec{B}$ satisfy

$$[A_i, A_j] = i\epsilon_{ijk} A_k \ , \quad [B_i, B_j] = i\epsilon_{ijk} B_k \ , \quad [A_i, B_j] = 0 \tag{14}$$

...in other words this is the group $SU(2) \times SU(2)$. Luckily we know all the irreducible representations of this group, as they are just labelled by two j quantum numbers:

$$R = (j_A, j_B) \tag{15}$$

a representation with dimension $(2j_A + 1)(2j_B + 1)$. You should think of where $A_i = a_i \otimes 1_B$ where a_i is a nontrivial $(2j_A + 1)$ matrix acting on the m_A indices, and 1_B is the trivial $(2j_B + 1)$ dimensional unit matrix

[1]The A_i and B_i I define are called N_i and $N_i^\dagger$ respectively by Srednicki in Ch. 33. This is misleading, since in general $\vec{A}$ and $\vec{B}$ ar enot hermitian conjugates of each other; obviously that can only happen when one has a hermitian representation for the K_i generators, which is manifestly not the case for the defining representation in eq. (9) – or for any finite dimensional representation. It *is* the case for the infinite dimensional representation in the Hilbert space, where Lorentz transformations have to be unitary (they conserve probability) and the generators hermitian.

acting on the m_B indices, and conversely $B_i = 1_A \otimes b_i$. The corresponding Lorentz transformation, from eq. (13) and eq. (12) is given by

$$D_{\{j_A,j_B\}} = e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{A}+i(\vec{\theta}-i\vec{\omega})\cdot\vec{B}} \tag{16}$$

Note the interesting feature that for the finite dimensional, hermitian representations of $SU(2)$ for A_i and B_i we have

$$D^*_{\{j_A,j_B\}}(\vec{\theta},\vec{\omega}) = e^{i(\vec{\theta}-i\vec{\omega})\cdot\vec{(}-A^*)+i(\vec{\theta}+i\vec{\omega})\cdot\vec{(}-B^*)} = D_{\{\bar{j}_B,\bar{j}_A\}} = D_{\{j_B,j_A\}} \tag{17}$$

where I used the facts that

(i) in general, if X_a is a representation R of a Lie algebra, then $-X_a^*$ is also a representation, and is called the conjugate representation (here denoted as $\bar{R}$);

(ii) the group $SU(2)$ only has real representations, so j and $\bar{j}$ are the same (up to a similarity transformation)...for example, for spin $\frac{1}{2}$ we have $-\sigma_i^* = \sigma_2\sigma_i\sigma_2$;

(iii) since conjugation flipped the relative sign between θ and ω, the roles of A and B are flipped.

Thus if ψ transforms according to the (j_A, j_B) representation, ψ^* is in the (j_B, j_A) representation.

A related observation which is useful is that for hermitian representations or $\vec{A}$ and $\vec{B}$, possible to define for all finite dimensional representations of SU(2),

$$D^\dagger_{\{j_A,j_B\}}(\vec{\theta},\vec{\omega}) = e^{-i(\vec{\theta}-i\vec{\omega})\cdot\vec{A}-i(\vec{\theta}+i\vec{\omega})\cdot\vec{B}} = D^{-1}_{\{j_B,j_A\}}(\vec{\theta},\vec{\omega}). \tag{18}$$

We now know all of the irreducible, finite dimensional representations of the Lorentz group. For example, assume we have a field ψ transforming as the $(\frac{1}{2}, 0)$ representation. Thus we have $\vec{a} = \frac{1}{2}\vec{\sigma}$ and $\vec{b} = 0$ and

$$D_L \equiv D_{(\frac{1}{2},0)}(\vec{\theta},\vec{\omega}) = e^{i(\vec{\theta}+i\vec{\omega})\cdot\vec{\sigma}/2} \ . \tag{19}$$

A rotation by angle θ_3 about the z axis corresponds to the matrix

$$D_{(\frac{1}{2},0)}(\theta_3) = e^{i\theta_3\sigma_3/2} = \begin{pmatrix} e^{i\theta_3/2} & 0 \\ 0 & e^{-i\theta_3/2} \end{pmatrix} \ . \tag{20}$$

A boost in the z direction with velocity parameter ω_3 corresponds to

$$D_{(\frac{1}{2},0)}(\omega_3) = e^{-\omega_3\sigma_3/2} = \begin{pmatrix} e^{-\omega_3/2} & 0 \\ 0 & e^{\omega_3/2} \end{pmatrix} \tag{21}$$

$\hat{z}$ rotations multiply the components of ψ by phases (a factor of -1 for a rotation by 2π), while a boost makes the upper component exponentially large, and the lower component exponentially small.

Similarly, if χ transforms according to the $(0, \frac{1}{2})$ representation, then $\chi \to D_R\chi$ where

$$D_R \equiv D_{(0,\frac{1}{2})} = e^{i(\vec{\theta}-i\vec{\omega})\cdot\vec{\sigma}/2} \ . \tag{22}$$

Rotations are the same as for ψ, but boosts have $\omega \to -\omega$.