# 4 The dynamical FRW universe

## 4.1 The Einstein equations

Einstein's equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{72}$$

relate the expansion rate $a(t)$ to energy distribution in the universe. On the left hand side is the Einstein tensor which can be determined from the metric, and is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \ . \tag{73}$$

where $R_{\mu\nu}$ and $R$ are the Ricci tensor and Ricci scalar respectively. On the right side of the equation is Newton's constant $G_N$ and the energy momentum tensor of matter, $T_{\mu\nu}$. The component $T_{00}$ tells us about the energy density, while $T_{ij}$ is related to the flow of material in the $j$ direction with nonzero momentum pointing in the $i$ direction.

For the FRW metric one finds

$$R_{00} = -3\frac{\ddot{a}}{a} \ , \qquad R_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right] g_{ij} \ , \qquad R_{0i} = 0 \ , \tag{74}$$

where $g_{ij}$ is the spatial part of the FRW metric. Noting that $g^i_j = \delta_{ij}$, it is easy to construct the Einstein tensor with one upper and one lower index,

$$\begin{aligned} G^0_0 &= 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \\ G^i_j &= \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \delta_{ij} \ . \end{aligned} \tag{75}$$

Our isotropy and homogeneity assumptions for the metric are justified if the energy-momentum tensor $T^\mu_\nu$ has a similar structure: independent of spatial coordinates, and with $T^i_j \propto \delta_{ij}$ with $T^0_i = T^i_0 = 0$. Throughout this course we will be treating matter to be close to an ideal fluid with energy density $\rho$ and pressure $p$, for which

$$T^\mu_\nu = \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix} \tag{76}$$

Einstein's equations then read:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G_N}{3}\rho \ , \tag{77}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G_N p \ . \tag{78}$$

It is useful to manipulate the equations to simplify the second one. By subtracting eq. (77) from eq. (78) we get an expression fro the cosmic acceleration:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) \ . \tag{79}$$

It may seem a little disturbing to find two differential equations to solve for one function, $a(t)$. However, we can manipulate our equations again to show that one combination of the two Einstein equations is simply equivalent to energy conservation. If we differentiate eq. (77) with respect to time, and then make use of both equations we find

$$\begin{aligned} \frac{8\pi G_N}{3}\dot{\rho} &= \left(\frac{\dot{a}}{a}\right)\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{k}{a^2}\right] \\ &= -8\pi G_N \left(\frac{\dot{a}}{a}\right)(\rho + p) \ , \end{aligned} \tag{80}$$

or

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + p) \ . \tag{81}$$

Note that this equation has no factor of $G_N$, and so is not telling us about gravity. In fact it can be rewritten as

$$\frac{d\left(a^3\rho\right)}{dt} = -p\frac{da^3}{dt} \tag{82}$$

which is the energy conservation equation

$$dE = -p\, dV \tag{83}$$

since a small cube of co-moving space has proper volume $dV = a^3(r^2 \sin\theta/\sqrt{1-kr^2})drd\theta d\phi$.

For most applications it is convenient to use the Friedmann equation eq. (77) as our first equation, and then either the equation for the acceleration eq. (79) or for energy conservation eq. (81) as our second independent equation.

#### 4.1.1 Summary

If we define

$$H = \left(\frac{\dot{a}}{a}\right) \ , \tag{84}$$

then our two equations to solve are:

$$\begin{aligned} H^2 + \tfrac{k}{a^2} &= \tfrac{8\pi G_N}{3}\rho \ , && \text{Friedmann equation} \\ \dot{\rho} &= -3H(\rho + p) \ , && \text{Energy conservation} \end{aligned} \tag{85}$$

### 4.2 Solutions for the equation of state $p = w\rho$

To proceed, we need to know the equation of state which relates $\rho$ and $p$. We will assume for now that the matter of the universe is a single material which satisfies

$$p = w\rho \ . \tag{86}$$

Various cases of particular interest are:

- *Non-relativistic, pressureless matter:* $w = 0$. If the universe is dominated by a dilute gas of non-relativistic particles, or a less dilute gas of weakly interacting particles, then its pressure is negligible, and we have $w = 0$.

- *Highly relativistic particles:* $w = 1/3$. When the universe is dominated by particles whose energy is so much greater than their mass that they can be treated as massless, then one finds $w = 1/3$. Note that for this special value, the energy momentum tensor eq. (76) is traceless, $T^\mu_\mu = 0$. There is a reason for this: if all your particles are massless, then they move at the speed of light and there is no special frame (unlike for massive particles, where there is a rest frame). But $T^\mu_\mu$ has dimensions of mass, and is invariant under Lorentz transformations. The only dimensionful quantity around is the energy of the particles, but that is not Lorentz invariant, so $T^\mu_\mu$ must vanish.
- *Cosmological constant (vacuum energy):* $w = -1$. In non-gravitational physics one can always add a constant to the Lagrangian without consequence. It shifts the value of the energy, but since only energy differences are measured, there is no innate definition of zero energy. However, gravity couples to energy, and shifting the Lagrangian by a constant shifts $T^\mu_\nu$. The constant term is called the cosmological constant; it is a divergent quantity in quantum field theory, and so naively one would think it should be very big, but as we will see, observations of our cosmos tell us it is very small. Since adding a constant to the Lagrangian is Lorentz invariant, it follows that the change in $T^\mu_\nu$ must be proportional to $g^\mu_\nu - \delta_{\mu\nu}$. Thus it must be that such a form of "matter" must have $-p = \rho$, or $w = -1$. Note that when $w < 0$, the pressure is negative, which implies tension...the material would want to contract in the absence of gravity.
- *Matter with* $w = -1/3$. Although there is no reason to expect matter with $w = -1/3$, it is interesting to note from eq. (79) that for $w < -1/3$, the expansion of the universe is accelerating ($\ddot{a} > 0$) while for $w > -1/3$, it is decelerating ($\ddot{a} < 0$).

From the equation for energy conservation eq. (85) we find $\dot{\rho} = -3H(1 + w)\rho$ or

$$\frac{d\rho}{\rho} = -3(1 + w)\frac{da}{a} \tag{87}$$

with solution

$$\rho \propto a^{-3(1+w)} \ . \tag{88}$$

Note that non-relativistic matter has $\rho \propto a^{-3}$, which accounts for dilution of the particles dues to the expansion of the volume they are in; for relativistic matter, $\rho \propto a^{-4}$, where the extra factor of $a^{-1}$ accounts for the redshift of the energy; and a cosmological constant has an energy density that remains constant as the universe expands, where the increase of energy comes from the universe expanding against the negative pressure.

Next we look at the Friedmann equation, eq. (85). A static solution is possible with $H = 0$ provided that $k \neq 0$ and

$$\frac{k}{a^2} = \frac{8\pi G_N}{3}\rho \ . \tag{89}$$

This solution appealed to Einstein on philosophical grounds before he learned of Hubble's discovery of cosmological expansion. It is not a stable configuration — small perturbations about this solution cause it to collapse or expand.

If $H \neq 0$ then we can rewrite the Friedmann equation as

$$\frac{k}{a^2} = H^2 (\Omega - 1) \ , \tag{90}$$

where

$$\Omega(t) = \frac{\rho(t)}{\rho_{\rm crit}(t)} \ , \qquad \rho_{\rm crit}(t) \equiv \frac{3H^2(t)}{8\pi G_N} \tag{91}$$

$$\rho_c \equiv \rho_{\rm crit}(t_0) = \frac{3H_0^2}{8\pi G_N} = 1.88h^2 \times 10^{-29}\,{\rm gm\,cm^{-3}} = 1.1\times 10^{-5}h^2 \ {\rm GeV\,cm^{-3}} \ .$$

In principle then we see how to tell what $k$ is: we measure the total energy density $\rho$ of the universe, we determine $\rho_c$ by measuring the Hubble constant and then by constructing the ratio we find $\Omega$. If $\Omega > 1$ we have $k = 1$, and if $\Omega < 1$ then $k = -1$, while if $\Omega = 1$, $k = 0$. Of course, if $a \gg 1$, the right hand side may be close to zero, even if $k \neq 0$; this is the idea behind the theory of inflation. As we will see later, we have evidence that $\Omega$ is very close or equal to one. That means that the curvature term in our universe is negligible.

Now suppose the universe contains all three types of energy: relativistic ($\rho_R$), nonrelativistic ($\rho_M$, "M" is for "matter"), and cosmological constant ($\rho_\Lambda$); we have seen that these scale as $a^{-4}$, $a^{-3}$, and $a^0$ respectively. Then the total energy density, as a function of the scale factor $a$, is given by

$$\begin{aligned} \rho &= \rho_R + \rho_M + \rho_\Lambda \\ &= \rho_c \left( \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda \right) \ . \end{aligned} \tag{92}$$

We see that the early universe (small $a$) will be radiation dominated, while the late universe is dominated by the cosmological constant, if nonzero, or matter if it is. Everything with the subscript 0 on $\rho$ and $H$ refers to today's value; the terms $\Omega_M$, $\Omega_R$ and $\Omega_\Lambda$ always refer to today's values. Using this formula we can find a useful formula for the evolution of $\Omega$. From eq. (89) we have

$$\Omega - 1 = (\Omega_0 - 1)\, \frac{a^2 H_0^2}{a_0^2 H^2} \ . \tag{93}$$

We can rewrite

$$\begin{aligned} \frac{H^2}{H_0^2} &= \frac{\rho_{\rm crit}(t)}{\rho_c} = \frac{\rho_{\rm crit}(t)}{\rho}\frac{\rho}{\rho_c} \\ &= \frac{1}{\Omega}\left( \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda \right) \ . \end{aligned} \tag{94}$$

Plugging the above result into the previous one, we get

$$\Omega - 1 = \frac{\Omega_0 - 1}{1 - \Omega_0 + \Omega_R \left(\frac{a_0}{a}\right)^2 + \Omega_M \left(\frac{a_0}{a}\right) + \Omega_\Lambda \left(\frac{a}{a_0}\right)^2} \ . \tag{95}$$

We see that in the early universe $(\Omega - 1) \propto (\Omega_0 - 1)(a/a_0)^2$, so that if $\Omega_0 \lesssim 1$ then $\Omega$ must have satisfied $\Omega = 1$ to high accuracy in the early universe, and will evolve to $\Omega = 0$ in the far future if $\Omega_\Lambda = 0$. In contrast, if $\Omega_0 > 1$, then we must also have had $\Omega = 1$ to high accuracy in the early universe, but in the future $\Omega$ will grow and eventually diverge.

If we set $k = 0$, it is simple to solve for for the evolution of the universe for a simple equation of state $p = w\rho$. By taking the time derivative of the Friedmann equation we get

$$2H\dot{H} = \frac{8\pi G_N}{3}\dot{\rho} = \frac{8\pi G_N}{3}\left[-3H(1+w)\rho\right] = -3H^3(1+w) \ . \tag{96}$$

Thus

$$\frac{dH}{H^2} = -\frac{3(1+w)}{2} dt \ , \tag{97}$$

with solution

$$H = \frac{2}{3(1+w)t} \ . \tag{98}$$

I have chosen the integration constant such that $H = \infty$ at $t = 0$. Writing $H = \dot{a}/a$ and integrating again, we find

$$a(t) = \begin{cases} a_0 e^{\pm\lambda t} & w = -1, \\ a_0 (t/t_0)^{2/3(1+w)} & w \neq -1 \end{cases} \tag{99}$$