2 Cosmology and GR

The first step toward a cosmological theory, following what we called the "cosmological principle" is to implement the assumptions of isotropy and homogeneity withing the context of general relativity (GR). I will assume that you are familiar with special relativity; some acquaintance with GR would be helpful but not necessary for this course. Good introductions can be found in the recent book *Gravity: An Introduction to Einstein's General Relativity* by James Hartle, or at Sean Caroll's web site.

General Relativity (GR) is the theory that controls the large scale evolution of the universe. There are two parts to the theory, which are generalizations the Poisson equation and Newton's law:

$$\nabla^2\phi = 4\pi G\rho \ , \tag{8}$$

$$\mathbf{a} = -\boldsymbol{\nabla}\phi \ . \tag{9}$$

The first of the above equations tells us how a mass distribution ρ gives rise to a gravitational potential ϕ, while the second tells us how a particle accelerates in the resulting potential. In GR, the Poisson equation eq. (8) is replaced by Einstein's equation

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ , \tag{10}$$

where $G_{\mu\nu}$ on the left hand side is called the Einstein tensor, and which describes the geometry of spacetime, while $T_{\mu\nu}$ on the right hand side is the energy-momentum tensor, describing the distribution of energy and momentum. $T_{\mu\nu}$ acts as a source for spacetime curvature, just as ρ serves as a source for the gravitational potential ϕ in Poisson's equation eq. (8).

Given a particular geometry, the trajectory that particles follow is given by a **geodesic**, which is path of shortest distance between two points, with the concept of distance suitably defined. Therefore the geodesic equation replaces Newton's law, eq. (9). I will begin by briefly discussing differential geometry and geodesics, then turning to Einstein's equations and some of their solutions.

2.1 The metric and and coordinate transformations

Consider some manifold in d dimensions, such as a curved two-dimensional surface, or four dimensional Minkowski space. We can refer to points on the surface by labeling them each with a set of d coordinates x^μ where $\mu = 1, \ldots, d$. The geometry of the manifold can be specified in terms of a **metric** $g_{\mu\nu}$ which allows us to compute the distance ds between the point at x^μ and one nearby at $x^\mu + dx^\mu$, where

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{11}$$

In general, $g_{\mu\nu}$ is a function of x^μ; it is symmetric, $g_{\mu\nu} = g_{\nu\mu}$. As is the convention in special relativity, an upper and a lower index which are identical are assumed to be summed over ("**contracted**").

If we change coordinate systems from x^μ to $\bar{x}^\mu$, then in terms of the new coordinates

$$ds^2 = g_{\mu\nu}\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\right)\left(\frac{\partial x^\nu}{\partial \bar{x}^\beta}\right) d\bar{x}^\alpha d\bar{x}^\beta = \bar{g}_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta \ , \tag{12}$$

so we see that in the new coordinate system, the metric is given by

$$\bar{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} \tag{13}$$

If we think of $g_{\mu\nu}$ as a matrix, then we can write the above transformation as a matrix equation

$$g(x) \to \bar{g}(\bar{x}) = \Lambda^T(\bar{x}) g(x(\bar{x}))\Lambda(\bar{x}) \ , \qquad \Lambda^\alpha{}_\beta \equiv \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \ . \tag{14}$$

We can also define the inverse of the metric, writing it with upper indices:

$$g^{\alpha\beta} g_{\beta\gamma} \equiv g^\alpha_\gamma = \begin{cases} 1 & \alpha = \gamma, \\ 0 & \text{otherwise.} \end{cases} \ . \tag{15}$$

In general, **tensors** are objects with upper and/or lower indices that behave in the following way under coordinate transformations $x \to \bar{x}$: for each lower index, the tensor $T^{\cdots}_{\cdots\alpha\cdots}$ gets multiplied by $\frac{dx^\alpha}{d\bar{x}^\beta}$ with the α's contracted; while for each upper index, the tensor $T^{\cdots\alpha\cdots}_{\cdots}$ gets multiplied by
$\frac{d\bar{x}^\beta}{dx^\alpha}$ with the α's contracted. The metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$ and the partial derivatives $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ are examples of tensors; the coordinate x^μ is *not* a tensor, but the differential dx^μ is.

Indices on tensors can be raised and lowered by means of the metric tensor. A string of tensors with some indices contracted is also a tensor. A tensor with no indices, or equivalently a string of tensors with all their indices contracted is a **scalar**, invariant under all coordinate transformations.

Here are some simple examples of metrics and coordinates in two dimensions:

	plane (Cartesian)	plane (polar)	2-sphere
x^μ	$\{x, y\}$	$\{r, \theta\}$	$\{\theta, \phi\}$
ds^2	$dx^2 + dy^2$	$dr^2 + r^2 d\theta^2$	$a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
$g_{\mu\nu}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$	$a^2\begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix}$
$g^{\mu\nu}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & r^{-2} \end{pmatrix}$	$a^{-2}\begin{pmatrix} 1 & \\ & \sin^{-2}\theta \end{pmatrix}$
x_μ	$\{x, y\}$	$\{r, r^2\theta\}$	$\{a^2\theta, a^2\phi\sin^2\theta\}$

(16)

One can easily arrive at the above metric in polar coordinates by making the change of variables

$$x = r\cos\theta \ , \quad y = r\sin\theta \tag{17}$$

and following the prescription in eq. (14). A more amusing change of variables is to take the sphere, expressed in terms of polar and azimuthal angles $\{\theta, \phi\}$, and to define $\rho = \sin\theta$. Then $d\rho = \cos\theta d\theta$, or $d\theta^2 = d\rho^2/(1 - \rho^2)$. Thus in the new variables $\{\rho, \phi\}$ we have

$$ds^2 = a^2\left(\frac{d\rho^2}{1-\rho^2} + \rho^2 d\phi^2\right) \ . \qquad \text{(sphere)} \tag{18}$$

This is still the metric for a sphere. Note that it looks similar to the metric for a plane with $r \equiv a\rho$.

2.2 Geodesics

Given a metric, one can compute the shortest path between two points; such paths are called **geodesics**. It is interesting to learn to compute them for two reasons: (i) objects moving in a gravitational field follow geodesics; (ii) computing geodesics is an efficient way to compute **Christoffel symbols**, an important object in differential geometry out of which on constructs **curvature**.

The equation for the geodesics can be simply derived using variational calculus. We parametrize a path $x^\mu(\lambda)$ by the parameter λ with, for example, $\xi^\mu(0) = x_i^\mu$ being the start of the path and $x^\mu(1) = x_f^\mu$ being the end of the path. Then we wish to stationarize the path length L:

$$0 = \delta L = \delta \int_{x_i^\mu}^{x_f^\mu} ds = \delta \int_0^1 \sqrt{\frac{d^2 s}{d\lambda^2}}\, d\lambda = \delta \int_0^1 \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda\ . \tag{19}$$

We can make our task slightly easier by noting that the path that extremizes the integral of $\sqrt{ds^2/d^2\lambda}$ also extremizes the integral of $\frac{1}{2}\, d^2 s/d\lambda^2$, so we solve

$$0 = \delta \int_0^1 \left(\tfrac{1}{2}\, g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) d\lambda\ , \tag{20}$$

which yields the "Lagrangian"

$$L = g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \tag{21}$$

which obeys the Euler Lagrange equation (where $\boldsymbol{g_{\mu\nu,\alpha} \equiv \partial g_{\mu\nu}/\partial x^\alpha}$)

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} \\ &= \frac{d}{d\lambda} \left(g_{\alpha\nu} \frac{dx^\nu}{d\lambda} \right) - \tfrac{1}{2} g_{\mu\nu,\alpha} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \\ &= g_{\alpha\nu,\beta} \frac{dx^\beta}{d\lambda} \frac{dx^\nu}{d\lambda} + g_{\alpha\nu} \frac{d^2 x^\nu}{d\lambda^2} - \tfrac{1}{2} g_{\mu\nu,\alpha} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\ , \end{aligned} \tag{22}$$

where $\boldsymbol{g_{\mu\nu,\alpha} \equiv \partial g_{\mu\nu}/\partial x^\alpha}$. By renaming dummy (contracted) indices, the above expression may be rewritten in the form

$$0 = \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\ , \tag{23}$$

$$\Gamma^\alpha_{\mu\nu} \equiv \tfrac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta} \right)\ . \tag{24}$$

$\Gamma^\alpha_{\mu\nu}$ is called a **Christoffel symbol**; it is symmetric in its lower indices,

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}\ . \tag{25}$$

One can show that under coordinate transformations iΓ transforms inhomogeneously, and is therefore not a tensor. Often the simplest way to compute the Christoffel symbols is not to use the formula eq. (24), but rather the variational equation eq. (20) in conjunction with eq. (23).

2.3 Two dimensional isotropic metrics

As an example it is interesting to compute the Christoffel symbols for the most general isotropic metric in two dimensions. We take as our coordinates $\{r, \theta\}$, in which case the most general line element is

$$ds^2 = A(r,\theta)dr^2 + B(r,\theta)d\theta^2 + C(r,\theta)d\theta dr \tag{26}$$

By making a change of variables, it is possible to cancel the mixed term proportional to $C(r,\theta)$. If we now insist that the metric be isotropic (independent of θ) then the line element takes the form

$$ds^2 = \tilde{A}(r)dr^2 + \tilde{B}(r)d\theta^2 \ . \tag{27}$$

I will assume that $\tilde{A}$ and $\tilde{B}$ are positive functions. Then we can make the change of variables $a^2\rho^2 = \tilde{B}(r)$, where a is a number of dimension length. We arrive at element

$$ds^2 = h(\rho)d\rho^2 + \rho^2 d\theta^2 \ , \tag{28}$$

where $h(\rho)$ is an arbitrary, dimensionless function. Since we do not want a conical singularity at the origin (if you roll a piece of paper into a cone, the tip of the cone is called a conical singularity) it follows that for a very small circle about the origin, we had better recover the flat space relation between the circumference C and the radius R of the circle: $C = 2\pi R$. With the above metric we find that for the circle $\{d\rho, 0 \le \theta < 2\pi\}$, the length of the radius is $R = h(0)d\rho$, while the length of the circumference is $C = 2\pi d\rho$. So we have the additional constraint

$$h(0) = 1 \ . \tag{29}$$

This is the most general isotropic metric in two dimensions.

Now let us compute the Christoffel symbols. From ds^2 we 'construct the "Lagrangian"

$$L = \left(h(\rho)\dot{\rho}^2 + \rho^2\dot{\theta}^2\right) \tag{30}$$

with the Euler-Lagrange equations

$$\begin{aligned} 0 &= \frac{d}{d\lambda}(2h\dot{\rho}) - \left(h'\dot{\rho}^2 + 2\rho\dot{\theta}^2\right) \ , \\ 0 &= \frac{d}{d\lambda}\left(2\rho^2\dot{\theta}\right) \ , \end{aligned} \tag{31}$$

which can be rewritten as

$$\ddot{\rho} + \frac{h'}{2h}\dot{\rho}^2 - \frac{\rho}{h}\dot{\theta}^2 = 0 \ , \tag{32}$$

$$\tag{33}$$

$$\ddot{\theta} + \frac{2}{\rho}\dot{\rho}\dot{\theta} = 0 \ . \tag{34}$$

By comparing eq. (34) with eq. (23), we find that the nonzero Christoffel symbols for this metric eq. (28) are just

$$\Gamma^1_{11} = \frac{h'}{2h} \ , \quad \Gamma^1_{22} = -\frac{\rho}{h} \ , \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho} \ . \tag{35}$$

2.4 Curvature

Next let us ask what the most general isotropic, homogeneous metric is in two dimensions. To see if the metric describes a homogeneous space, we need to define something that is supposed to remain constant over the manifold. The quantity should not be coordinate dependent, or what looks constant in in set of coordinates does not in another, and by "homogeneity" we refer to an intrinsic property of the space independent of coordinate choice. The scalar curvature such a quantity.

Suppose you started walking East along the earth's equator holding an arrow in you hand pointed East. After walking a quarter of the way around the world, you turn at right angles heading North, but you are careful to not turn the arrow; as you march North, the arrow is pointing at 90° to your right.. When you reach the North Pole, you make a 90° left turn and head South, again careful not to rotate the arrow, which now points directly behind you as you proceed back to the equator. When you reach the equator, you are back at your starting point; however the arrow which you were careful to never rotate relative to your path (**parallel transport**) is now pointing North, whereas it was pointing East when you began your journey. This rotation of the arrow is a result of the **curvature** of the Earth; nothing analogous occurs when tracing a triangular path on a flat surface.

By considering infinitesimal closed paths on a manifold and the effect on the orientation of arrows (vectors) one can make precise a definition of curvature. One arrives at the definition of three useful and related tensors for describing curvature, The **Riemann tensor**, the **Ricci tensor**, the **Ricci scalar**, and the **Einstein tensor**. Their definitions are

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} &= \frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\gamma\epsilon}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\epsilon}\Gamma^{\epsilon}_{\beta\gamma} && \textbf{Riemann tensor} \\ R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} &= \frac{\partial\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\gamma}_{\alpha\gamma}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} && \textbf{Ricci tensor} \\ R &= R^{\alpha}{}_{\alpha} && \textbf{Ricci scalar} \end{aligned} \tag{36}$$

The Riemann tensor has some simplifying symmetries:

- Antisymmetry: $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$;
- Symmetry: $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$;
- A cyclic property: $R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$.

As a result of these symmetries, one finds that the Riemann tensor in d dimensions has in general $C_d = d^2(d^2-1)/12$ independent components. For $d = 2, 3, 4$ dimensions we find $C_2 = 1$, $C_3 = 6$ and $C_4 = 10$. Metrics with special symmetry can have far fewer independent components.

The Ricci scalar R is something that must be constant everywhere on our two dimensional manifold if it is to describe a homogeneous space.

2.5 Two dimensional isotropic and homogeneous metrics

So as a warm-up for four dimensional cosmology, let's compute the Ricci scalar for the isotropic metric eq. (28), demand it be constant, and determine the allowed functions $h(\rho)$ in the metric eq. (28). Since we are in two dimensions, there is only one independent component of the Riemann tensor. For the two-dimensional isotropic metric with non-vanishing Christoffel symbols given by eq. (35), the Riemann tensor is given by

$$R^1{}_{212} = \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{21} = \frac{\rho h'}{2h^2} . \tag{37}$$

Then, using the symmetries of the Riemann tensor,

$$R^2{}_{121} = g^{22} g_{11} R^1{}_{212} = \frac{h'}{2\rho h} . \tag{38}$$

It follows that the Ricci tensor is

$$R_{11} = R^2{}_{121} = \frac{h'}{2\rho h} , \quad R_{22} = R^1{}_{212} = \frac{\rho h'}{2h^2} , \quad R_{12} = R_{21} = 0 , \tag{39}$$

or

$$R_{\alpha\beta} = \frac{h'}{2\rho h^2} \begin{pmatrix} h & \\ & \rho^2 \end{pmatrix} = \frac{h'}{2\rho h^2} g_{\alpha\beta} . \tag{40}$$

Finally this implies that the Ricci scalar is given by

$$R = R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta} = \frac{h'}{\rho h^2} . \tag{41}$$

Requiring that R equal a constant c everywhere gives us the differential equation for the function $h(\rho) : h'/h^2 = c\rho$, with solution

$$h = \frac{1}{1 - c\rho^2/2} , \tag{42}$$

where I have implemented the boundary condition on h, eq. (29). We can now rescale $\rho \to \sqrt{2/|c|}\rho$, to obtain the metric

$$ds^2 = a^2 \left(\frac{d\rho^2}{1 - k\rho^2} + \rho^2 d\theta^2 \right) , \tag{43}$$

with $k = c/|c| = 0, \pm 1$, and $a^2 = 2/|c|$. The parameter a has dimension of length and is called the scale factor; the curvature $R = c$ is given by $R = 2k/a^2$. Note that the sign of the curvature appears as the parameter k: a flat space corresponds to $k = 0$, and the metric is just that for the plane in polar coordinates; if $k = 1$ we have constant positive curvature and get the metric for the sphere that we derived in eq. (18). The case $k = -1$, a surface with constant negative curvature, is the surface AdS_2, two dimensional anti-de Sitter space. It can be described as a surface embedded in three dimensional Minkowski space, namely the surface satisfying

$$z^2 - x^2 - y^2 = 0 \tag{44}$$

in a three dimensional space with line element

$$ds^2 = dx^2 + dy^2 - dz^2 . \tag{45}$$

As we will see, the metric eq. (43) is similar to what we will encounter when describing the isotropic, homogeneous universe.