Group Theory Problem Set #4 Due Thursday **2/9/06**

Two of the following problems will be graded and are worth five points each.

Note: I will be out of town the week of 2/13-2/18. The class will be taught by Dr. Chris Lee.

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1. Compute the following Clebsch-Gordon coefficients:
   (a) $\langle \frac{3}{2}, -\frac{1}{2}; 1, 1 | \frac{3}{2}, \frac{1}{2} \rangle$
   (b) $\langle 1, 0; \frac{1}{2}, \frac{1}{2} | \frac{3}{2}, \frac{1}{2} \rangle$

2. Georgi, 4A.

3. Georgi, 4B

4. Assuming exact isospin symmetry, find a relation between the scattering amplitudes (at a given energy) for the inelastic scattering processes

   - $\pi^- p \to K^0 \Sigma^0$
   - $\pi^- p \to K^+ \Sigma^-$
   - $\pi^+ p \to K^+ \Sigma^+$

   where the pions (π^-, π^0, π^+) and the sigma baryons $(\Sigma^-, \Sigma^0, \Sigma^+)$ are isospin triplets, while the nucleons (n, p) and the kaon mesons (K^0, K^+) are isospin doublets; all these particles are arranged in order of increasing I_3. You can assume that the interactions preserve isospin. (Hint: there will be two independent amplitudes, corresponding to arranging the initial (and final) particles in an $I = 3/2$ or an $I = 1/2$ multiplet. Therefore there will be a relationship between three processes.) Can you relate the three scattering cross sections for the above processes? (Cross sections are proportional to the absolute value squared of the amplitudes.)

5. Consider the Hamiltonian for a two dimensional simple harmonic oscillator:

$$\hat{H} = \hbar\omega \sum_{i=1,2} \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right) , \tag{1}$$

   where the $\hat{a}_i$ and $\hat{a}_i^\dagger$ operators obey the commutation relations

$$[a_i, a_j] = [a_i^\dagger a_j^\dagger] = 0 , \qquad [a_i, a_j^\dagger] = \delta_{ij} . \tag{2}$$

(a) What are the energies and the number of degenerate states of the n^{th} energy level (counting the ground state as level $n = 0$)?

(b) Show that the four operators

$$\hat{N} = \sum_i a_i^\dagger a_i \ , \qquad \hat{T}_b = \sum_i a_i^\dagger \left(\frac{\sigma_b}{2}\right)_{ij} a_j \ , \ b = 1, 2, 3, \qquad (3)$$

all commute with $\hat{H}$.

(c) Compute the commutators $[\hat{N}, \hat{T}_a]$ and $[\hat{T}_a, \hat{T}_b]$. What group G do these four operators generate?

(d) Compute the commutators $[\hat{N}, \hat{a}_i^\dagger]$ and $[\hat{T}_a, \hat{a}_i^\dagger]$. Use this information to classify your energy eigenstates as transforming under irreducible representations of the group G, and thereby explain the degeneracy number of each energy level.