Group Theory Problem Set #2 Due Thursday 1/26/06

The following two problems are worth five points each; both will be graded.

1. Consider the following subset of matrices of $SU(3)$ matrices (e.g. three dimensional unitary matrices with determinant one):

$$A_{pq} = \begin{pmatrix} \omega^p & 0 & 0 \\ 0 & \omega^q & 0 \\ 0 & 0 & \omega^{-p-q} \end{pmatrix}, \quad B_{pq} = \begin{pmatrix} 0 & \omega^p & 0 \\ 0 & 0 & \omega^q \\ \omega^{-p-q} & 0 & 0 \end{pmatrix}, \quad C_{pq} = \begin{pmatrix} 0 & 0 & \omega^p \\ \omega^q & 0 & 0 \\ 0 & \omega^{-p-q} & 0 \end{pmatrix}, \quad (1)$$

where $\omega = e^{i2\pi/3}$ and p and q each take on any of the values 0, 1, or 2. It is easy to check that these matrices form a group of order 27, called $\Delta(27)$.

(a) Is the above representation of $\Delta(27)$ reducible or not?

(b) Show that in general, if $D_R(g)$ is a representation of a group G, then the conjugate matrices $D_R^*(g)$ form a representation as well. We call this the conjugate representation $\bar{R}$, with $D_{\bar{R}}(g) \equiv D_R^*(g)$.

(c) Is the above representation for $\Delta(27)$ equivalent to its conjugate representation?

(d) How many conjugacy classes does this group have?

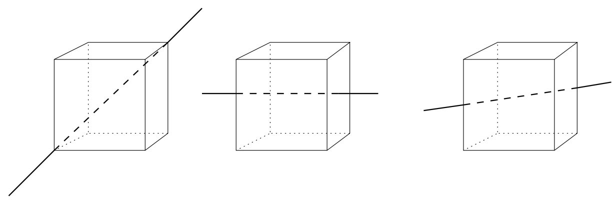
(e) What are the dimensions of the irreducible representations of this group?

2. The octahedral group O consists of all of the rotational symmetries of the cube (but not reflection symmetries). Its character table is given by

O	e	$8C_3$	$3C_2$	$6C_4$	$6C_2'$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

(2)

The different classes are quite easy to understand: the $8C_3$ class corresponds to the rotations by $\pm 2\pi/3$ about the 3-fold symmetry axis passing through opposite corners on the cube (there are four such axes, two types of rotations each, which adds up to eight). The $3C_2$ class corresponds to the π rotations about the any of the three axes passing through the centers of opposing faces; the $6C_4$ class corresponds to rotations by $\pm\pi/2$ about those same three axes. Finally the $6C_2'$ class corresponds to π rotations about the six axes joining the midpoints of opposite edges.



Examples of axes for the classes $8C_3$*;* $3C_2$ *and* $6C_4$*;* $6C_2'$*.*

(a) How does the vector (x, y, z) decompose into irreducible representations of O? (The action of O symmetries on this vector defines a 3-dimensional representation; compute its characters).

(b) Suppose you have a molecule possessing O symmetry and you want to compute electric dipole transitions $\langle R_1|\vec{d}|R_2\rangle$ between states transforming as irreducible representations $R_{1,2}$ of O, where $\vec{d}$ is the electric dipole operator. What are the selection rules? That is, for which R_1 and R_2 can this matrix element be nonzero? Use the fact that $\vec{d}$ transforms under rotations like a three vector, i.e., just like (x, y, z).

(c) The magnetic moment operator $\vec{m}$ transforms like an axial vector. Are the selection rules for magnetic moment transitions going to be the same or different than those for electric dipole moment transitions?