Group Theory Problem Set #1 Due Thursday 1/19/06

The following problems are worth five points each; two will be graded.

1. Which of the following describe groups, and which do not? Of the ones which do describe groups, which are Abelian? Justify your answers.

	e	**a**	**b**	**c**	**d**
e	**e**	**a**	**b**	**c**	**d**
a	**a**	**d**	**c**	**e**	**b**
b	**b**	**c**	**a**	**d**	**e**
c	**c**	**e**	**d**	**b**	**a**
d	**d**	**b**	**e**	**a**	**c**

Table 1

	e	**a**	**b**	**c**	**d**
e	**e**	**a**	**b**	**c**	**d**
a	**a**	**b**	**c**	**d**	**e**
b	**b**	**d**	**a**	**e**	**c**
c	**c**	**e**	**d**	**b**	**a**
d	**d**	**c**	**e**	**a**	**b**

Table 2

   (a) Table 1 above;

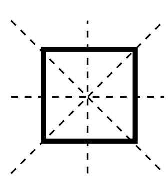
   (b) Table 2 above;

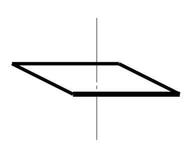
   (c) The four matrices $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, under matrix multiplication, where σ_0 is the 2×2 unit matrix, and $\sigma_{1,2,3}$ are the Pauli matrices;

   (d) The rational numbers under division;

   (e) Unitary 3×3 matrices U with $\det U = 1$, under matrix multiplication;

2. Consider the group G of transformations that leave a square invariant (it's called C_{4v}). It consists of $\{e, C_4, C_2, C_4^3, \sigma_x, \sigma_y, \sigma_{d1}, \sigma_{d2}\}$, corresponding to the identity (e), clockwise rotations by $\pi/2$, π and $3\pi/2$ about the center (C_4, C_2, C_4^3 respectively); reflections about the x and y axes (σ_x, σ_y); and reflections about the diagonal axes passing through the first and second quadrants (σ_{d1}, σ_{d2}).





(a) Construct the multiplication table for the group. (Hint: label the corners of the square, and see how the different group elements g permute them. Remember that a product $g_1 g_2$ means "act first with g_2, then with g_1").

	e	C_4	C_2	C_4^3	σ_x	σ_y	σ_{d1}	σ_{d2}
e								
C_4								
C_4^3								
σ_x								
σ_y								
σ_{d1}								
σ_{d2}								

(b) Construct the conjugacy classes for this group.

(c) How many irreducible representations exist for this group, and what are their dimensions?

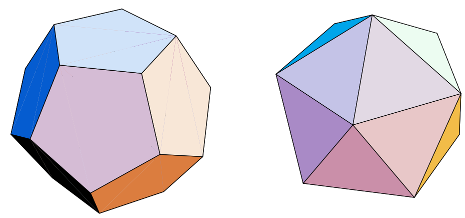
(d) Determine all of the 1-dimensional representations for this group.

(e) Consider how the unit vectors in the plane $\hat{x}$ and $\hat{y}$ transform under the group, and thereby construct a 2-dimensional representation of the group (call it E). Is this representation irreducible?

(f) Write down the character table for this group, and decompose $E \otimes E$ into irreducible representations.

3. Show that the only Abelian group of order 74 is the cyclic group Z_{74}. (Hint: think about its cyclic subgroups.)

4. The icosahedral group is the symmetry group of the dodecahedron and the icosahedron, pictured below:



The character table for this group is given by

I	e	$12C_5$	$12C_5^2$	$20C_3$	$15C_2$
A	1	1	1	1	1
T_1	3	$-2\cos 144°$	$-2\cos 72°$	0	-1
T_2	3	$-2\cos 72°$	$-2\cos 144°$	0	-1
G	4	-1	-1	1	0
H	5	0	0	-1	1

(1)

From the notation and the figures, you should be able to relate the C_5, C_3 and C_2 as five-fold, three-fold, and two-fold rotational symmetries of these solids.

There is a subgroup of I called D_5 (a dihedral group) consisting of the ten group elements $\{e, C_5, C_5^2, C_5^3, C_5^4, C_2^{(1)}, \ldots, , C_2^{(5)}\}$, where C_5 is a rotation by $2\pi/5$ about one of the five-fold symmetry axes of the dodecahedron (call it the z-axis), and the $C_2^{(i)}$ the rotations by π about the five two-fold symmetry axes in the $x-y$ plane. The character table for D_5 is given by

$\mathbf{D_5}$	e	$2C_5$	$2C_5^2$	$5C_2$
A_1	1	1	1	1
A_2	1	1	1	-1
E_1	2	$2\cos 144°$	$2\cos 72°$	0
E_2	3	$2\cos 72°$	$2\cos 144°$	0

(2)

(a) Decompose the irreducible representations of I into irreducible representations of the D_5 subgroup. (this is analagous to saying that a four-vector of Lorentz symmetry transforms as a scalar plus a 3-vector under the subgroup of spatial rotations, such as $p_\mu = \{E, \vec{p}\}$).

(b) Suppose a Hamiltonian takes the form $H = H_I + \lambda H_{D_5}$ where H_I is invariant under I, while H_{D_5} is less symmetric, only respecting the subgroup D_5. Suppose that when $\lambda = 0$, there is an energy eigenstate which transforms as the H representation of the I, and is 5-fold degenerate. How do the energy levels of these five states split up as λ is turned on (that is, what are their exact degeneracies)?