The Gell-Mann–Okubo mass formula

I was unable to finish my lecture last Friday, and as you will have substitute lecturers this week, I decided to finish off the lecture in print.

I showed that the octet pseudoscalars and baryons could be represented by the following tensors:

$$\Phi^i_j = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} , \tag{1}$$

$$B^i_j = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} , \qquad \overline{B}^i_j = \begin{pmatrix} \frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Sigma^-} & \overline{\Xi^-} \\ \overline{\Sigma^+} & -\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Xi^0} \\ \bar{p} & \bar{n} & -\frac{2\overline{\Lambda}}{\sqrt{6}} \end{pmatrix} , \tag{2}$$

Under an $SU(3)$ transformation U, these fields transform as $\Phi \to U^\dagger \Phi U$, $B \to U^\dagger B U$, $\overline{B} \to U^\dagger \overline{B} U$. We can therefore write down a Hamiltonian for these particles at rest which is $SU(3)$ invariant, and which describes nonzero particle masses:

$$H_0 = \frac{1}{2}\mu^2 \mathrm{Tr}\Phi^2 + M_0 \mathrm{Tr}\overline{B}B \ . \tag{3}$$

The masses of the eight baryons are M_0 and the masses squared of the eight mesons are μ^2.

Now we introduce $SU(3)$ breaking. We know that whatever breaks $SU(3)$ to a good approximation preserves $SU(2)\times U(1)$. Therefore we looked for a nontrivial $SU(3)$ representations which contain an $SU(2)\times U(1)$ singlet. We saw that the octet was the smallest such representation. In general we can write an octet as $M^i_j = m_a (T_a)^i_j$. For this to be invariant under $SU(2)\times U(1)$, we require that only m_8 be nonzero. So we write the symmetry breaking parameter as $M^i_j = m_8 (T_8)^i_j$. Note that $[T_i, T_8] = 0$ for $i = 1, 2, 3, 8$ which is why we know that $m_8 (T_8)^i_j$ does not break $SU(2)\times U(1)$.

The symmetry breaking part of the Hamiltonian, to linear order in m_8 (which we treat as a perturbation) takes the form

$$H' = \frac{\alpha}{2}\mathrm{Tr}\Phi^2 M + \beta \mathrm{Tr}\overline{B} M B + \gamma \mathrm{Tr}\overline{B} B M \ . \tag{4}$$

Now we need only plug in the matrices and compute. However we can make life simpler if we write

$$M = m_8 T_8 = m_0 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + m_s \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} , \tag{5}$$

where $m_0 = m_8/\sqrt{12}$, and $m_s = -2m_8/\sqrt{12}$. Now we can ignore the contrributions from m_0 since they are $SU(3)$ preserving and can be absorbed into the Hamiltonian H_0 in Eq. (3). So we find that just keeping the m_s term we get for the meson masses squared from $H_0 + H'$

$$m_\pi^2 = \mu^2 \ , \quad m_K^2 = \mu^2 + \frac{1}{2}\alpha m_s \ , \quad m_\eta^2 = \mu^2 + \frac{2\alpha}{3} m_s \ , \tag{6}$$

from which we find the relation

$$m_\eta^2 = \frac{4m_k^2 - m_\pi^2}{3} \ . \tag{7}$$

Plugging in $m_\pi = 140$ MeV, $m_k = 495$ MeV, we get $m_\eta = 565$ Mev, which is off from the experimental value of 547 MeV by 3.5%.

For the baryons we get from H' (only keeping the m_s term in M):

$$m_N = M_0 + \beta m_s \ , \quad m_\Sigma = M_0 \ , \quad m_\Lambda = M_0 + \frac{2(\alpha+\beta)m_s}{3} \ , \quad m_\Xi = M_0 + \alpha m_s \ , \tag{8}$$

where m_N signifies the nucleon mass (p or n). We have four masses (m_N, n_Σ, m_Λ and m_Ξ) and three unknowns (M_0, αm_s, and βm_s) and so we have a prediction:

$$m_\Lambda = \frac{2}{3}(m_N + m_\Xi) - \frac{1}{3} m_\Sigma \ . \tag{9}$$

From the experimental masses $m_N = 940$ MeV, $m\Sigma = 1190$ MeV, and $m_\Xi = 1320$ MeV, we get the prediction $m_\Lambda = 1110$ MeV, which is 0.5% off the experimental value of 1115 MeV!