Physics 507, Winter 2006

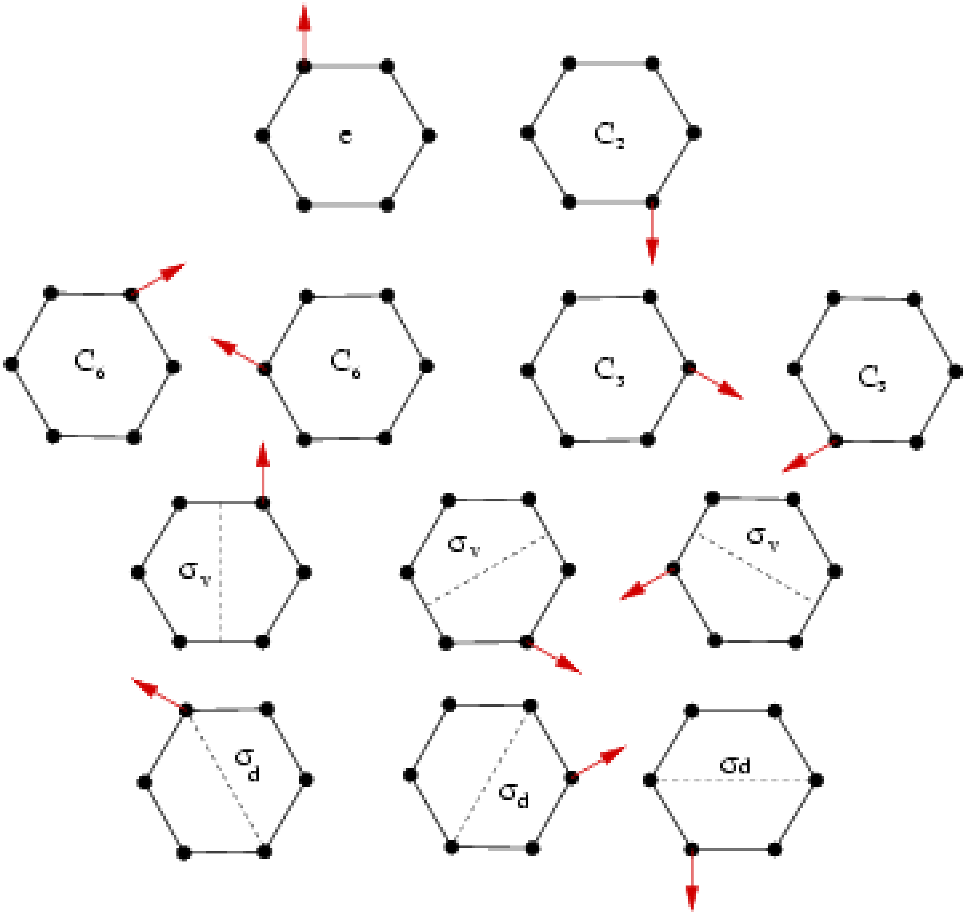
**Final Exam Solutions**

1.

| $\mathbf{C}_{6v}$ | $e$ | $2C_6$ | $2C_3$ | $C_2$ | $3\sigma_v$ | $3\sigma_d$ |
|---|---|---|---|---|---|---|
| $B_1$ | 1 | −1 | 1 | −1 | 1 | −1 |
| $B_2$ | 1 | −1 | 1 | −1 | −1 | 1 |

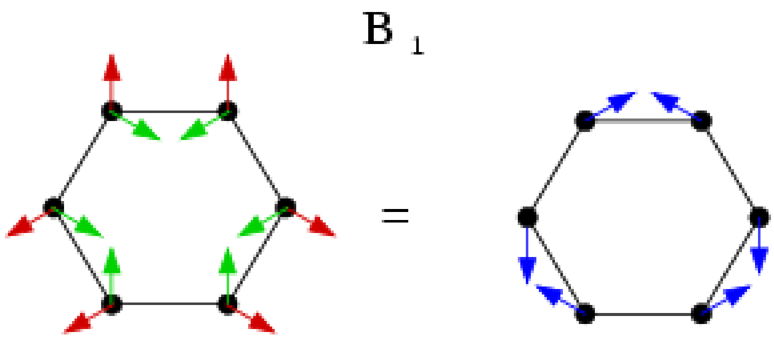
(1)

Since a picture is requested for this problem, we can solve it in pictures. I consider a particular displacement of one of the masses (see the picture labelled "e" below), and consider the action of the group on this configuration, pictured below (each new configuration is labelled by the equivalency class of the transformation that gave rise to it).



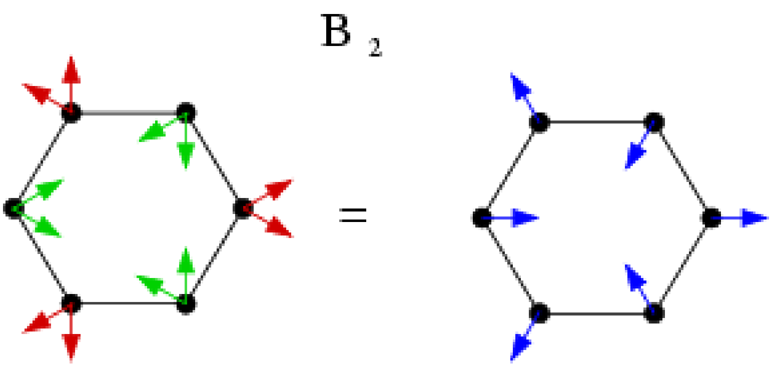
(a) *Making use of a projection operator, draw an accurate sketch indicating the initial displacement corresponding to the vibrational normal mode which transforms as the* $\mathbf{B}_1$ *representation of* $C_{6v}$.

Now we just add the configurations in the previous picture, weighted by the appropriate character of the $B_1$ representation...this is just vector addition, so a weight $+1$ means to add the little red vector in the previous diagram, while a weight $-1$ means to add $(-1)\times$ the little red vector. Below I have shown the hexagon with all of the positive contributions from the $e$, $C_3$ and $\sigma_v$ classes as red arrows, and all the negative contributions from the $C_6$, $C_2$ and $\sigma_d$ classes as green arrows. Adding the red and green vectors gives the shape of the $B_1$ mode, shown with blue arrows. (The normalization of the displacement or projection operator are irrelevant for this excercise.)




(b) *Same as above, for the* $\mathbf{B}_2$ *normal mode.*

Here the positive contributions (red arrows) are from the $e$, $C_3$, and $\sigma_d$ equivalency classes, while the negative contributions are from $C_6$, $C_2$ and $\sigma_v$ (green) which sum up to give the $B_2$ normal mode (blue):



**2.** *Consider a world which has fermions B and $\overline{B}$ (which I will refer to as "baryons") which transform in the adjoint representation of an approximate $SU(4)$ symmetry. This symmetry is broken by a small parameter M analogous to the quark mass matrix in the real world. M transforms as singlet ⊕ adjoint under the $SU(4)$ symmetry (i.e., $M \to UMU^\dagger$, where U is an $SU(4)$ matrix in the defining representation). Suppose M takes the form*

$$M = \begin{pmatrix} m_1 & & & \\ & m_1 & & \\ & & m_1 & \\ & & & m_2 \end{pmatrix} \tag{2}$$

*with $m_1 \neq m_2$.*

(a) *If the only symmetry breaking parameter in this world is the matrix M, what is the exact symmetry $H \subset SU(4)$ of the theory ?*

$H = SU(3) \times U(1)$; the generators for this symmetry are

$$T_a = \left(\begin{array}{c|c} \frac{\lambda_a}{2} & \\ \hline & 0 \end{array}\right) , \qquad Q = \frac{1}{\sqrt{24}} \left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & 1 & \\ \hline & & & -3 \end{array}\right) ,$$

where $T_a$ generate the $SU(3)$ with $a = 1, \ldots, 8$ and $\lambda_a$ being the Gell-Mann matrices; and $Q$ being the $U(1)$ generator. These are the nine $SU(4)$ generators which commute with $M$.

(b) *What are the irreducible H representations which appear in the decomposition of the $SU(4)$ "baryon" adjoint under the exact symmetry H ? (This is analogous to the real-world determination of the $SU(2)_I \times U(1)_Y$ representations in the baryon octet).*

$$15 \to 8_0 \oplus 3_4 \oplus 3_{-4} \oplus 1_0 \ ,$$

where the subscript equals $\sqrt{24} \times Q$. The overall normalization of the $U(1)$ is not important for this problem.

(c) *Invent a convenient parametrization of the matrix $B^i_j$.*

I will call the octet $O_a$ $(a = 1, \ldots 8)$, the triplet $T^i$ $(i = 1, 2, 3)$, the anti-triplet $T^c_i$ ("c" for "conjugate"), and the singlet $S$. The natural parametrization is then

$$B \;=\; \left(\begin{array}{ccc|c} & \frac{1}{\sqrt{2}} O_a(\lambda_a)_{ij} + \frac{1}{\sqrt{12}} S\delta_{ij} & & \begin{matrix} T^1 \\ T^2 \\ T^3 \end{matrix} \\ \hline T^c_1 & T^c_2 & T^c_3 & -\frac{3}{\sqrt{12}} S \end{array}\right) ,$$

$$\overline{B} = \left( \begin{array}{ccc|c} & \frac{1}{\sqrt{2}}\,\overline{O}_a(\lambda_a)_{ij} + \frac{1}{\sqrt{12}}\,\overline{S}\delta_{ij} & & \begin{matrix} \overline{T}^{c1} \\ \overline{T}^{c2} \\ \overline{T}^{c3} \end{matrix} \\ \hline \overline{T}_1 & \overline{T}_2 & \overline{T}_3 & -\frac{3}{\sqrt{12}}\,\overline{S} \end{array} \right) \tag{3}$$

where $i, j = 1, 2, 3$. Note that $T^c \neq \overline{T}$, just as $\Xi^- \neq \overline{P}$ in the real world. $B$ is normalized so that $\mathrm{Tr}\,\overline{B}B = \overline{O}_a O_a + \overline{T}_i T^i + \overline{T}^{c\,i} T^c_i + \overline{S}S$.

(d) *Considering the "baryon" mass splitting to linear order in $M$, what is the analogue of the Gell-Mann - Okubo formula for the "baryon" masses in this world?*

The $SU(4)$ symmetric masses will be given by

$$H_0 = m_0 \mathrm{Tr}\,\overline{B}B = m_0\left(\overline{O}_a O_a + \overline{T}_i T^i + \overline{T}^{c\,i} T^c_i + \overline{S}S\right)$$

As in the real world, the mass splittings will be governed at linear order in $M$ by two operators:

$$\begin{aligned} H_1 &= c_1 \mathrm{Tr}\,\overline{B}MB + c_2 \mathrm{Tr}\,\overline{B}BM \\ &= (c_1 m_1 + c_2 m_1)\overline{O}_a O_a + (c_1 m_1 + c_2 m_2)\overline{T}T \\ &\quad + (c_1 m_2 + c_2 m_1)\overline{T}^c T^c + (c_1 + c_2)\left(\frac{3m_1 + 9m_2}{12}\right)\overline{S}S\,. \end{aligned} \tag{4}$$

Therefore the "baryon" are

$$\begin{aligned} M_8 &= m_0 + (c_1 + c_2)m_1\,, \\ M_3 &= m_0 + c_1 m_1 + c_2 m_2\,, \\ M_{\bar{3}} &= m_0 + c_1 m_2 + c_2 m_1\,, \\ M_1 &= m_0 + (c_1 + c_2)\left(\frac{3m_1 + 9m_2}{12}\right)\,, \end{aligned} \tag{5}$$

from which we deduce the relation

$$M_1 = \frac{3M_3 + 3M_{\bar{3}} - 2M_8}{4}\,. \tag{6}$$

which is analogous to the Gell-Mann – Okubo prediction for the $\Lambda$ mass in terms of the $N$, $\Sigma$ and $\Lambda$ masses.

**3.** *Consider the following matrices (in direct product matrix notation)*

$$\sigma_a \otimes 1,\ 1 \otimes \eta_a,\ \sigma_a \otimes \eta_b\ ,$$

*where $\sigma_a$ and $\eta_a$ are Pauli matrices with $a, b = 1, 2, 3$. Count the matrices to check that you understand the notation — you should find 15 of them; each of the matrices is $4 \times 4$. You can convince yourself that these matrices form a closed algebra. They are normalized to give $\mathrm{Tr}\, X_a X_b = 4\delta_{ab}$. Choose your Cartan generators ($H_i$) to be given by $\sigma_3 \times 1$, $1 \otimes \eta_3$ and $\sigma_3 \otimes \eta_3$.*

(a) *Find the weights of this representation.*

The four weights are the eigenvalues of $H_1 = \sigma_3 \otimes 1$, $H_2 = 1 \otimes \eta_3$ and $H_3 = \sigma_3 \otimes \eta_3$:

$$\{1,1,1\}, \quad \{1,-1,-1\}, \quad \{-1,1,-1\}, \quad \{-1,-1,1\}\ .$$

Note that the eigenvalue of $H_3$ is always the product of the eigenvalues of $H_1$ and $H_2$.

(b) *How many root vectors should there be? Find them all.*

There are 15 generators, of which three are in the Cartan subalgebra, and so there must be 12 roots. There are 12 differences between the weights for this representation, so they must be the roots:

$$\pm\{2,\pm 2,0\}, \quad \pm\{0,2,\pm 2\}, \quad \pm\{2,0,\pm 2\},$$

(c) *Give the positive roots, following Georgi's convention for positivity in §8.1, where a positive vector is one whose first nonzero component is positive.*

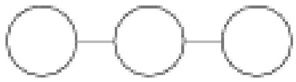
$$\{2,\pm 2,0\}\ , \quad \{2,0,\pm 2\}\ , \quad \{0,2,\pm 2\}\ .$$

(d) *Find the simple roots from among the positive roots found above. Number them so that $\alpha_1 > \alpha_2 > \alpha_3$ using Georgi's positivity condition from §19.1 — where a positive vector is one whose last nonzero component is positive.*

$$\alpha_1 = \{0,2,2\}\ , \quad \alpha_2 = \{2,-2,0\}\ , \quad \alpha_3 = \{0,2,-2\}\ .$$

(e) *Draw the Dynkin diagram for the group. What is the name of the group?*

We have $|\alpha_1|^2 = |\alpha_2|^2 = |\alpha_3|^2 = 8$, and $\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = -4$, while $\alpha_1 \cdot \alpha_3 = 0$. Therefore $\theta_{12} = \theta_{23} = 120°$ while $\theta_{13} = 0$. The Dynkin diagram is



which is recognized as that for $SU(4)$ or $SO(6)$.

(f) *Compute the Cartan matrix $A_{ij}$*

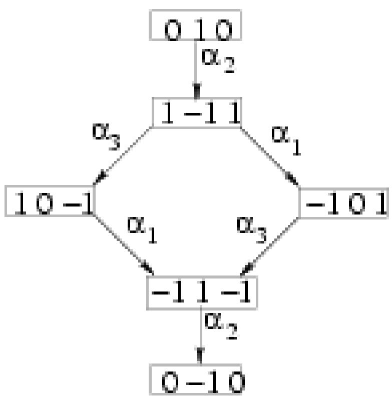
$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{|\alpha_j|^2} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}_{ij}$$

(g) *Find the fundamental weights.*

The fundamental weights satisfy $\frac{2\alpha_i \cdot \mu_j}{|\alpha_i|^2} = \delta_{ij}$. By inspection, the solution is:

$$\mu_1 = \{1,1,1\} \ , \quad \mu_2 = \{2,0,0\} \ , \quad \mu_3 = \{1,1,-1\} \ .$$

(h) *Find the weights for the representation whose Dynkin indices are $(0,1,0)$. Is this a real representation or not? Why?* I follow the familiar construction using the Cartan matrix:




We see that we have a 6-dimensional antisymmetric tensor representation of $SU(4)$, , or alternatively, the adjoint of $SO(6)$. The weights are read off the diagram, and describe the corners of a cube of edge length $= 4$:

$$\begin{aligned} \pm\mu_2 &= \pm\{2,0,0\} \ , \\ \pm(\mu_1 - \mu_2 + \mu_3) &= \pm\{0,2,0\} \ , \end{aligned}$$

$$\pm(\mu_1 - \mu_3) \quad = \quad \pm\{0, 0, 2\} \ . \tag{7}$$

The representation is real, as can be seen from the Dynkin diagram ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$ equals its complement for $SU(4)$). We can also see it from the weights: under conjugation, a weight goes to minus itself, and real representations are those where both signs of every weight appear, as is the case here.