Physics 507, Winter 2006

**Final Exam**

This exam is due by 12 noon, Monday March 13. Please return it either directly to me (C433), or if I'm not in my office, to any of the secretaries in the INT office (C411, near my office).

There are three problems on this exam, equally weighted. Do them all. If you can't complete a problem do what you can, and make your work legible!

You may refer to Georgi's book, your class notes and solution sets — nothing else (no other texts, nothing on the web, and you are not allowed to discuss the problems with anyone). If you have questions, you are welcome to email me, dbkaplan@phys.

1. In class I used the method of projection operators to find the vibrational normal modes of four masses in a plane connected by equivalent springs in the shape of a square. This system possessed a $C_{4v}$ symmetry. I wrote the eight numbers representing the displacements of the four masses from the equilibrium position in the form of a column vector $\xi(t)$; I then chose some particularly convenient initial displacement $\xi_0$, and acted on it with the projection operators $P_R$, where $R$ were the irreducible representations of $C_{4v}$ and $P_R$ were the projection operators

$$P_R = \frac{d_R}{{}^oG} \sum_g \chi_R(g)^* D^R(g) \ ,$$

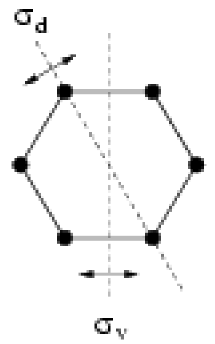
where $d_R$ is the dimension of $R$ and $\chi_R(g)$ is the character corresponding to the group element $g$.

Consider a similar problem where six masses in a plane are connected by springs in the shape of a hexagon, with a $C_{6v}$ symmetry. The character table for $C_{6v}$ is

| $\mathbf{C}_{6v}$ | $e$ | $2C_6$ | $2C_3$ | $C_2$ | $3\sigma_v$ | $3\sigma_d$ |
|---|---|---|---|---|---|---|
| $A_1$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_2$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ |
| $B_1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $B_2$ | 1 | $-1$ | 1 | $-1$ | $-1$ | 1 |
| $E_1$ | 2 | 1 | $-1$ | $-2$ | 0 | 0 |
| $E_2$ | 2 | $-1$ | $-1$ | 2 | 0 | 0 |

(1)

The $C_6$, $C_3$ and $C_2$ equivalence classes correspond to rotations by angles $\pm\frac{2\pi}{6}$, $\pm\frac{2\pi}{3}$, and $\pi$ respectively. The $\sigma_v$ correspond to reflections about the three axes connecting midpoints of opposite sides of the hexagon, while the $\sigma_d$ correspond to reflections about the three axes connecting opposite vertices of the hexagon.




(a) Making use of a projection operator, draw an accurate sketch indicating the initial displacement corresponding to the vibrational normal mode which transforms as the $\mathbf{B}_1$ representation of $C_{6v}$.

(b) Same as above, for the $\mathbf{B}_2$ normal mode.

**2.** Consider a world which has fermions $B$ and $\overline{B}$ (which I will refer to as "baryons") which transform in the adjoint representation of an approximate $SU(4)$ symmetry. This symmetry is broken by a small parameter $M$ analogous to the quark mass matrix in the real world. $M$ transforms as singlet $\oplus$ adjoint under the $SU(4)$ symmetry (i.e., $M \to UMU^\dagger$, where $U$ is an $SU(4)$ matrix in the defining representation). Suppose $M$ takes the form

$$M = \begin{pmatrix} m_1 & & & \\ & m_1 & & \\ & & m_1 & \\ & & & m_2 \end{pmatrix} \tag{2}$$

with $m_1 \neq m_2$.

(a) If the only symmetry breaking parameter in this world is the matrix $M$, what is the exact symmetry $H \subset SU(4)$ of the theory ?

(b) What are the irreducible $H$ representations which appear in the decomposition of the $SU(4)$ "baryon" adjoint under the exact symmetry $H$? (This is analogous to the real-world determination of the $SU(2)_I \times U(1)_Y$ representations in the baryon octet).

(c) Invent a convenient parametrization of the matrix $B^i_j$.

(d) Considering the "baryon" mass splitting to linear order in $M$, what is the analogue of the Gell-Mann - Okubo formula for the "baryon" masses in this world?

**3.** Consider the following matrices (in direct product matrix notation)

$$\sigma_a \otimes 1,\ 1 \otimes \eta_a,\ \sigma_a \otimes \eta_b\ ,$$

where $\sigma_a$ and $\eta_a$ are Pauli matrices with $a, b = 1, 2, 3$. Count the matrices to check that you understand the notation — you should find 15 of them; each of the matrices is $4 \times 4$. You can convince yourself that these matrices form a closed algebra. They are normalized to give $\mathrm{Tr}\, X_a X_b = 4\delta_{ab}$. Choose your Cartan generators ($H_i$) to be given by $\sigma_3 \times 1$, $1 \otimes \eta_3$ and $\sigma_3 \otimes \eta_3$.

(a) Find the weights of this representation.

(b) How many root vectors should there be? Find them all.

(c) Give the positive roots, following Georgi's convention for positivity in §8.1, where a positive vector is one whose *first* nonzero component is positive.

(d) Find the simple roots from among the positive roots found above. Number them so that $\alpha_1 > \alpha_2 > \alpha_3$ using Georgi's positivity condition from §13.1 — where a positive vector is one whose *last* nonzero component is positive.

(e) Draw the Dynkin diagram for the group. What is the name of the group?

(f) Compute the Cartan matrix $A_{ij}$

(g) Find the fundamental weights.

(h) Find the weights for the representation whose Dynkin indices are $(0, 1, 0)$. Is this a real representation or not? Why?

End of exam...have a good Spring break!