Physics 324, Fall 2001 **Solutions to problem set #5** Fri. 11/16/01

Problems chosen to be graded are marked by ⋆

⋆ **(1) a)**

$$\begin{aligned} [\hat{A}\hat{B},\hat{C}] &= \hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B} \\ &= \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B} \\ &= \hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B} \end{aligned} \tag{1}$$

b) Define $[\hat{p},\hat{x}^n]=-[\hat{x}^n,\hat{p}]\equiv\hat{C}_n$. Then, using the result of (a) we have

$$\begin{aligned} \hat{C}_n &= -[\hat{x}^n,\hat{p}]=-[\hat{x}\hat{x}^{n-1},\hat{p}] \\ &= -\hat{x}[\hat{x}^{n-1},\hat{p}]-[\hat{x},\hat{p}]\hat{x}^{n-1} \\ &= \hat{x}\hat{C}_{n-1}-i\hbar\hat{x}^{n-1}\ . \end{aligned} \tag{2}$$

This means that if we know $\hat{C}_{n-1}$ we can compute C_n. But we know C_1 so we can compute them all:

$$\hat{C}_1=[\hat{p},\hat{x}]=-i\hbar,\quad \hat{C}_2=\hat{x}\hat{C}_1-i\hbar\hat{x}=-2i\hbar\hat{x}\ ,\quad \hat{C}_3=\hat{x}\hat{C}_2-i\hbar\hat{x}^2=-3i\hbar\hat{x}^2\ , \tag{3}$$

or in general

$$\hat{C}_n=[\hat{p},\hat{x}^n]=-ni\hbar\hat{x}^{n-1}=-i\hbar\frac{d}{d\hat{x}}\hat{x}^n\ . \tag{4}$$

Similar reasoning gives

$$[\hat{x},\hat{p}^n]=+ni\hbar\hat{p}^{n-1}=i\hbar\frac{d}{d\hat{p}}\hat{p}^n\ . \tag{5}$$

c) If $V(\hat{x})$ has a Taylor expansion

$$V(\hat{x})=\sum_{n=0}^{\infty}c_n\hat{x}^n$$

then

$$\begin{aligned} [\hat{p},V(\hat{x})] &= \sum_{n=0}^{\infty}c_n[\hat{p},\hat{x}^n]=-i\hbar\sum_{n=0}^{\infty}c_n\frac{d}{d\hat{x}}\hat{x}^n \\ &= -i\hbar\frac{d}{d\hat{x}}\sum_{n=0}^{\infty}c_n\hat{x}^n=-i\hbar\frac{dV(\hat{x})}{d\hat{x}}\ . \end{aligned} \tag{6}$$

d)

$$
\begin{aligned}
[\hat{a}^\dagger \hat{a}, \hat{a}] &= \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = 0 - \hat{a} = -\hat{a} \ , \\
[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] &= \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger + 0 = \hat{a}^\dagger \ .
\end{aligned} \tag{7}
$$

e)

$$
\begin{aligned}
[\hat{L}_z, \hat{L}^2] &= [\hat{L}_z, (\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2)] \\
&= [\hat{L}_z, \hat{L}_x^2] + [\hat{L}_z, \hat{L}_y^2] + [\hat{L}_z, \hat{L}_z^2] \\
&= \Big(\hat{L}_x[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_x\Big) + \Big(\hat{L}_y[\hat{L}_z, \hat{L}_y] + [\hat{L}_z, \hat{L}_y]\hat{L}_y\Big) + 0 \\
&= (i\hat{L}_x\hat{L}_y + i\hat{L}_y\hat{L}_x) + (-i\hat{L}_y\hat{L}_x - i\hat{L}_x\hat{L}_y) \\
&= 0 \ .
\end{aligned} \tag{8}
$$

⋆ **(2)** **a)** We know that

$$
i\hbar \frac{d}{dt}|\psi, t\rangle = \hat{H}|\psi, t\rangle \ , \tag{9}
$$

which implies the conjugate equation

$$
-i\hbar \frac{d}{dt}\langle\psi, t| = \langle\psi, t|\hat{H} \ . \tag{10}
$$

Then, the chain rule gives

$$
\begin{aligned}
\frac{d}{dt}\langle\psi, t|\hat{O}|\psi, t\rangle &= \left(\frac{d}{dt}\langle\psi, t|\right)\hat{O}|\psi, t\rangle + \langle\psi, t|\frac{d\hat{O}}{dt}|\psi, t\rangle + \langle\psi, t|\hat{O}\left(\frac{d}{dt}|\psi, t\rangle\right) \\
&= \frac{1}{i\hbar}\langle\psi, t|[\hat{O}, \hat{H}]|\psi, t\rangle + \langle\psi, t|\frac{d\hat{O}}{dt}|\psi, t\rangle \ ,
\end{aligned} \tag{11}
$$

which gives the desired formula, provided that the operator $\hat{O}$ does not depend on time. The operators $\hat{x}$ and $\hat{p}$ are examples of such time independent operators.

b) Using the above result, and the results of problem (1), we have

$$
\begin{aligned}
\frac{d}{dt}\langle\psi, t|\hat{x}|\psi, t\rangle &= \frac{1}{i\hbar}\langle\psi, t|[\hat{x}, \hat{H}]|\psi, t\rangle \\
&= \frac{1}{i\hbar}\langle\psi, t|[\hat{x}, \left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)]|\psi, t\rangle \\
&= \frac{1}{i\hbar}\langle\psi, t|[\hat{x}, \frac{\hat{p}^2}{2m}]|\psi, t\rangle = \frac{1}{m}\langle\psi, t|\hat{p}|\psi, t\rangle \ .
\end{aligned} \tag{12}
$$

Next, we find

$$
\begin{aligned}
\frac{d}{dt}\langle\psi,t|\hat{p}|\psi,t\rangle &= \frac{1}{i\hbar}\langle\psi,t|[\hat{p},\hat{H}]|\psi,t\rangle \\
&= \frac{1}{i\hbar}\langle\psi,t|[\hat{p},\left(\frac{\hat{p}^2}{2m}+V(\hat{x})\right)]|\psi,t\rangle \\
&= \frac{1}{i\hbar}\langle\psi,t|[\hat{p},V(\hat{x})]|\psi,t\rangle \\
&= -\langle\psi,t|\frac{dV(\hat{x})}{d\hat{x}}|\psi,t\rangle \ . \qquad (13)
\end{aligned}
$$

(3) a)

$$
\begin{aligned}
1 = \langle\psi,0|\psi,0\rangle &= |N|^2(\langle 3| + i\langle 4|)(|3\rangle - i|4\rangle) \\
&= |N|^2\left(\langle 3|3\rangle - i\langle 3|4\rangle + i\langle 4|3\rangle + \langle 4|4\rangle\right) \\
&= |N|^2(1+0+0+1) = 2|N|^2 \ . \qquad (14)
\end{aligned}
$$

so

$$N = \frac{1}{\sqrt{2}}$$

normalizes the state.

b) First of all we need the fact that

$$|\psi,t\rangle = N\left(e^{-iE_3t/\hbar}|3\rangle - ie^{-iE_4t/\hbar}|4\rangle\right) \ , \qquad (15)$$

where $E_n = \hbar\omega(n+1/2)$. Next, as in class we write

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\ \hat{x} + i\sqrt{\frac{1}{2\hbar m\omega}}\ \hat{p} \ , \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\ \hat{x} - i\sqrt{\frac{1}{2\hbar m\omega}}\ \hat{p} \ , \qquad (16)$$

which is inverted to give

$$\hat{x} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) \ , \quad \hat{p} = -\frac{i}{2}\sqrt{2\hbar m\omega}\left(\hat{a}-\hat{a}^\dagger\right) \ . \qquad (17)$$

We can then write

$$
\begin{aligned}
\langle\psi,t|\hat{x}|\psi,t\rangle &= \frac{|N|^2}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(e^{iE_3t/\hbar}\langle 3| + ie^{+iE_4t/\hbar}\langle 4|\right)\left(\hat{a}+\hat{a}^\dagger\right)\left(e^{-iE_3t/\hbar}|3\rangle - ie^{-iE_4t/\hbar}|4\rangle\right) \\
&= \frac{|N|^2}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(-ie^{i(E_3-E_4)t/\hbar}\langle 3|\hat{a}|4\rangle + ie^{-i(E_3-E_4)t/\hbar}\langle 4|\hat{a}^\dagger|3\rangle\right) \ , \qquad (18)
\end{aligned}
$$

where I have kept the only nonvanishing amplitudes. These can be computed to be:

$$\langle 3|\hat{a}|4\rangle = \sqrt{4}\langle 3|3\rangle = 2 \ , \qquad \langle 4|\hat{a}^\dagger|3\rangle = \langle 3|\hat{a}|4\rangle^* = 2 \ . \tag{19}$$

So finally, since $(E_4 - E_3) = \hbar\omega$ and $i(e^{i\omega t} - e^{-i\omega t}) = -2\sin\omega t$,

$$\langle \psi, t|\hat{x}|\psi, t\rangle = -2|N|^2\sqrt{\frac{2\hbar}{m\omega}}\sin\omega t = -\sqrt{\frac{2\hbar}{m\omega}}\sin\omega t \tag{20}$$

Note that the expectation value of the position oscillates about $\langle x\rangle = 0$ with the classical frequency ω.

c) We can repeat the above steps, using the formula eq. (17) for $\hat{p}$:

$$\begin{aligned}\langle \psi, t|\hat{p}|\psi, t\rangle &= -|N|^2\frac{i}{2}\sqrt{2\hbar m\omega}\left(e^{iE_3t/\hbar}\langle 3| + ie^{+iE_4t/\hbar}\langle 4|\right)(\hat{a} - \hat{a}^\dagger)\left(e^{-iE_3t/\hbar}|3\rangle - ie^{-iE_4t/\hbar}|4\rangle\right) \\ &= -|N|^2\frac{i}{2}\sqrt{2\hbar m\omega}\left(-ie^{i(E_3-E_4)t/\hbar}\langle 3|\hat{a}|4\rangle - ie^{-i(E_3-E_4)t/\hbar}\langle 4|\hat{a}^\dagger|3\rangle\right) \\ &= -2|N|^2\sqrt{2\hbar m\omega}\ \cos\omega t = -\sqrt{2\hbar m\omega}\ \cos\omega t \ . \end{aligned} \tag{21}$$

We can check to see if our answers obey the results of problem (2):

$$\begin{aligned}\frac{d}{dt}\langle \psi, t|\hat{x}|\psi, t\rangle &= \frac{d}{dt}\left(-\sqrt{\frac{2\hbar}{m\omega}}\sin\omega t\right) = -\omega\sqrt{\frac{2\hbar}{m\omega}}\cos\omega t = -\frac{1}{m}\sqrt{2\hbar m\omega}\cos\omega t \\ &= \frac{1}{m}\langle \psi, t|\hat{p}|\psi, t\rangle \end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{d}{dt}\langle \psi, t|\hat{p}|\psi, t\rangle &= \frac{d}{dt}\left(-\sqrt{2\hbar m\omega}\ \cos\omega t\ \right) = \omega\sqrt{2\hbar m\omega}\sin\omega t = m\omega^2\sqrt{\frac{2\hbar}{m\omega}}\sin\omega t \\ &= -m\omega^2\langle \psi, t|\hat{x}|\psi, t\rangle \ . \end{aligned} \tag{23}$$

Using the fact that $dV(\hat{x})/d\hat{x} = d(k\hat{x}^2/2)/d\hat{x} = k\hat{x} = m\omega^2\hat{x}$ (where I used $\omega = \sqrt{k/m}$, we see that indeed

$$\frac{d}{dt}\langle \psi, t|\hat{p}|\psi, t\rangle = -\langle \psi, t|\frac{dV(\hat{x})}{d\hat{x}}|\psi, t\rangle \ . \tag{24}$$