Physics 324, Fall 2001 **Solutions to problem set #4** Fri. 11/9/01

Problems chosen to be graded are marked by ⋆

---

(1) *Gasiorowicz 4-4*

We have $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ Since all the $u_n(x)$ are even or odd under $x \to a - x$, $|u_n(x)|^2$ is always even under $x \to a - x$, and so $\langle x\rangle = a/2$. Then

$$\begin{aligned}
\langle x^2\rangle &= \frac{2}{a}\int_0^a dx\, x^2 \sin^2(n\pi x/a) \\
&= \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} dy\, y^2 \sin^2 y \\
&= \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 n\pi \left(\frac{n^2\pi^2}{6} - \frac{1}{4}\right) \\
&= \frac{a^2}{6}\left(2 - \frac{3}{n^2\pi^2}\right) . \qquad (1)
\end{aligned}$$

This implies that

$$\Delta x = a\frac{\sqrt{1 - 6/(n\pi)^2}}{\sqrt{12}} .$$

Then it is easy to see that $\langle p\rangle = 0$, while

$$\begin{aligned}
(\Delta p)^2 = \langle p^2\rangle &= -\hbar^2 \frac{2}{a}\int_0^a \sin(n\pi x/a)\frac{d^2}{dx^2}\sin(n\pi x/a) \\
&= \hbar^2 \frac{2}{a}\left(\frac{n\pi}{a}\right)^2 \int_0^a \sin^2(n\pi x/a) \\
&= \left(\frac{\hbar n\pi}{a}\right)^2 . \qquad (2)
\end{aligned}$$

Therefore

$$\Delta p \Delta x = \hbar\frac{n\sqrt{\pi^2 - 6/n^2}}{\sqrt{12}} \geq (1.13)\frac{\hbar}{2} .$$

(2) *Gasiorowicz 4-6*

a) No, the wavefunction will spread out.

b) The probability P_n that the particle is in the n^{th} energy eigenstate is

$$P_n = |\langle n|\psi\rangle|^2 = \left|\int_{-a/2}^{a/2} dx\,\langle n|x\rangle\langle x|\psi,0\rangle\right|^2 = \left|\int_{-a/2}^{a/2} dx\,u_n^*(x)\psi(x)\right|^2 . \qquad (3)$$

For this geometry, we have

$$u_1(x) = \sqrt{\frac{2}{a}}\cos(\pi x/a)\ , \qquad u_2(x) = \sqrt{\frac{2}{a}}\sin(2\pi x/a)\ .$$

(Sketch these wave functions and convince yourself that they look exactly like those in figure 4-2, only shifted to the right by $a/2$.) So we get

$$P_1 = \left(\frac{2}{a}\right)^2 \left|\int_{-a/2}^{0} dx\,\cos(\pi x/a)\right|^2 = \frac{4}{\pi^2}\ ,$$

$$P_2 = \left(\frac{2}{a}\right)^2 \left|\int_{-a/2}^{0} dx\,\sin(2\pi x/a)\right|^2 = \frac{4}{\pi^2}\ . \qquad (4)$$

So the particle has equal probability, $P_1 = P_2 = 4/\pi^2 = 0.41$ for being in either the ground state or the first excited state. That is, there is an equal probability of $4/\pi^2$ that an energy measurement will yield E_1 or E_2.

(3)