2 problems chosen to be graded are marked by a $\star$ and are worth 40 points each; the tutorial questions are worth an additional 20 points total.

$\star$ **Gasiorowicz 2-12** Several ways to do this. Note that the question is dealing with standard wave properties, and not wave-particle duality, so there will be no $\hbar$ in the final answer. One way: a property of Fourier transforms is $\Delta x \Delta k \gtrsim 1$ (this is the same as $\Delta x \Delta p \gtrsim \hbar$, if you use the de Broglie relations). Here, $\Delta x = c\Delta t$, so $\Delta k \gtrsim 1/(c\Delta t)$. To relate to $\Delta\lambda$, note that $k = 2\pi/\lambda$, so a small change in k by dk and a small change in λ by $d\lambda$ are related by $dk = (-2\pi/\lambda^2)d\lambda$. Taking the absolute value, and calling $|dk| = \Delta k$, $\Delta\lambda = |d\lambda|$ we get

$$\Delta\lambda = \frac{\lambda^2}{2\pi}\Delta k \gtrsim \frac{\lambda^2}{2\pi c\Delta t} = \frac{(6000\,\mathrm{A})^2}{2\pi(3\times 10^{18}\,\mathrm{A/s})10^{-9}\,\mathrm{s}} = 1.9\times 10^{-3}\,\mathrm{A}\ . \tag{1}$$

Note that since $\Delta\lambda \ll \lambda$, it was justified to use differentials $d\lambda$ to represent $\Delta\lambda$. If instead of this method you used the uncertainty relations for $\Delta x\Delta p$ or $\Delta E\Delta t$, that should have given you the same answer.

Gasiorowicz 3-6 Given:

$$\psi = \sqrt{\frac{2a^3}{\pi}}\frac{1}{x^2+a^2}\ , \qquad \int_{-\infty}^{\infty} dx\,|\psi|^2 = 1\ , \tag{2}$$

$$\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+a^2} = \frac{\pi}{a}\ , \quad \phi(p) = \sqrt{\frac{a}{\hbar}}e^{-a|p|/\hbar}\ . \tag{3}$$

Note that $|\psi(x)|^2$ and $|\phi(p)|^2$ are even functions of x and p respectively, so

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\,x|\psi(x)|^2 = 0\ , \qquad \langle p\rangle = \int_{-\infty}^{\infty} dp\,p|\phi(p)|^2 = 0\ . \tag{4}$$

Then

$$\begin{aligned}\langle x^2\rangle &= \int_{-\infty}^{\infty} dx\,x^2|\psi|^2 = -a^2\int_{-\infty}^{\infty} dx\,|\psi|^2 + \int_{-\infty}^{\infty} dx\,(x^2+a^2)|\psi|^2 \\ &= -a^2 + \frac{2a^3}{\pi}\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+a^2} = -a^2 + \frac{2a^3}{\pi}\frac{\pi}{a} = a^2\ ,\end{aligned} \tag{5}$$

and

$$\begin{aligned}\langle p^2\rangle &= \int_{-\infty}^{\infty} dp\, p^2|\phi(p)|^2 = 2\frac{a}{\hbar}\int_0^{\infty} dp\, p^2 e^{-2ap/\hbar} \\ &= \frac{2a}{\hbar}\left(\frac{\hbar}{-2}\right)^2 \frac{d^2}{da^2}\left(\int_0^{\infty} dp\, e^{-2ap/\hbar}\right) = \frac{\hbar a}{2}\frac{d^2}{da^2}\left(\frac{\hbar}{2a}\right) = \frac{\hbar}{2a^2}\ . \qquad (6)\end{aligned}$$

Therefore

$$\Delta x = \sqrt{\langle x^2\rangle} = a\ , \quad \Delta p = \sqrt{\langle p^2\rangle} = \frac{\hbar}{a\sqrt{2}}\ , \qquad (7)$$

and $\Delta x \Delta p = \hbar/\sqrt{2} > \hbar/2$, in keeping with the Heisenberg uncertainty relation.

Gasiorowicz 3-8 Again, there are many ways to solve this problem, some harder than others. I will show you a nice way and a very slick way.

First the nice way. In the momentum representation, $\hat{p} = p$ and $\hat{x} = -\frac{\hbar}{i}\frac{d}{dp}$, where $\hat{p}$ and $\hat{x}$ are operators. Therefore

$$\begin{aligned}e^{i\hat{p}a/\hbar}\hat{x}e^{-i\hat{p}a/\hbar} &= e^{ipa/\hbar}\left(-\frac{\hbar}{i}\frac{d}{dp}\right)e^{-ipa/\hbar} \\ &= e^{ipa/\hbar}\left(ae^{-ipa/\hbar} + e^{-ipa/\hbar}\left(-\frac{\hbar}{i}\frac{d}{dp}\right)\right) \\ &= a - \frac{\hbar}{i}\frac{d}{dp} = a + \hat{x}\ . \qquad (8)\end{aligned}$$

The second, very slick, way is to define an operator which is a function of the number a:

$$\hat{F}(a) = e^{i\hat{p}a/\hbar}\hat{x}e^{-i\hat{p}a/\hbar}\ . \qquad (9)$$

Note that $\hat{F}(0) = \hat{x}$. Now take the derivative with respect to a:

$$\begin{aligned}\frac{d\hat{F}(a)}{da} &= e^{i\hat{p}a/\hbar}\left(\frac{i\hat{p}}{\hbar}\hat{x} + \hat{x}\frac{-i\hat{p}}{\hbar}\right)e^{-i\hat{p}a/\hbar} \\ &= e^{i\hat{p}a/\hbar}\frac{i}{\hbar}[\hat{p},\hat{x}]e^{-i\hat{p}a/\hbar} = e^{i\hat{p}a/\hbar}e^{-i\hat{p}a/\hbar} = 1\ , \qquad (10)\end{aligned}$$

where I used $[\hat{p},\hat{x}] = \hbar/i$ (eq. 3-38) and that fact that $\hat{p}$ commutes with itself. So now we see that $\hat{F}(a)$ satisfies

$$\frac{d\hat{F}(a)}{da} = 1\ , \quad \hat{F}(0) = \hat{x} \qquad (11)$$

which is a first order differential equation with boundary condition, with the unique solution

$$\hat{F}(a) = \hat{x} + a\ . \qquad (12)$$

⋆ **Gasiorowicz 3-9** We have the wave function $\psi(\theta)$, $\theta \in [-\pi, \pi]$ with the boundary condition $\psi(\pi) = \psi(-\pi)$. This could be the wave function for a particle living on a circle of wire, with θ being the angular coordinate around the circle. The boundary condition on ψ then is equivalent to saying that ψ is continuous and single-valued on the circle.

We have $\hat{L} = \frac{\hbar}{i}\frac{d}{d\theta}$ (L is the angular momentum around the circle, but you don't need to know that for this problem). Then

$$\langle L \rangle = \int_{-\pi}^{\pi} d\theta\, \psi^* \hat{L} \psi = \frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta\, \psi^* \frac{d}{d\theta} \psi \ . \tag{13}$$

It follows then that

$$\langle L \rangle^* = -\frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta\, \psi \frac{d}{d\theta} \psi^* \ . \tag{14}$$

Integrating by parts, we get

$$\langle L \rangle^* \ = \ -\frac{\hbar}{i} |\psi(\theta)|^2 \Big|_{-\pi}^{\pi} + \frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta \left(\frac{d}{d\theta} \psi \right) \psi^* = \langle L \rangle \ , \tag{15}$$

since the first term vanishes, given that $\psi(\pi) = \psi(-\pi)$. That is the same as saying that the boundary term in the integration by parts vanishes, since a circle has no boundary. Therefore $\hat{L}$ has a real expectation value, and is a "hermitian" operator.