Problems chosen to be graded are marked by $\star$

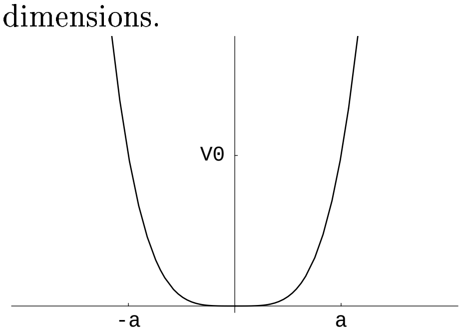
$\star$ **(1)** **(a)** We can take $\Delta x = L$ and therefore $\Delta p \gtrsim \hbar/L$. We have $p \gtrsim \Delta p$ so that the particle's energy satisfies $E \gtrsim (\Delta p)^2/2m \gtrsim \hbar^2/(2mL^2)$. Therefore $E_0 \simeq \hbar^2/(2mL^2)$ is a good estimate of the ground state energy (the minimum energy the particle can have).

(b) The velocity is $v = p/m \gtrsim \Delta p/m \gtrsim \hbar/(mL)$, so $v_{min} \simeq \hbar/(mL)$.

(c) **(i)** $m = .51$ MeV$/c^2$, $L = 1.0\times10^{-7}$ cm. Use $\hbar = 6.6\times10^{-22}$ MeV-s, $c = 3.0\times10^{10}$ cm/s: $v_{min} \simeq (6.6\times10^{-22})(3.0\times10^{10})^2/((.51)(1.0\times10^{-7})) = 1.1\times10^7$ cm/s, roughly one third the speed of light.

(ii) $m = 1.0$ gm, $L = 10$ cm. Use $\hbar = 1.0\times10^{-27}$ erg-s. Then $v_{min} \simeq (1.0\times10^{-27})/((1.0)(10.)) = 1.0\times10^{-28}$ cm/s. That is awfully slow – in the age of the universe (10 billion years) it would travel 3×10^{-11} cm at that rate.

(2) **(a)** $E = p^2/2m+V$. Write $p = \hbar/\Delta x$, $x = \Delta x$, so that $E = \hbar^2/(2m\Delta x^2)+V_0(\Delta x/a)^4$. Now minimize with respect to Δx. The real solutions to the minimization yield $\Delta x^2 = \hbar a/\sqrt{2mV_0})$ and $E_{min} = \sqrt{2V_0\hbar^2/(a^2m)}$. You should be able to check that this has the right dimensions.



(b) A good sign that a particle is relativistic is when $E \gtrsim mc^2$. Using the energy from the above, this occurs when $\sqrt{2V_0\hbar^2/(a^2m)} \gtrsim mc^2$, or $a \lesssim \sqrt{2V_0\hbar^2/(c^4m^3)}$.

(3) **(a)** We wish to solve

$$1 = \int_{-\infty}^{\infty} dp\,|\phi(p)|^2 = N^2 \int dp\, e^{-(p-\bar{p})^2/p_0^2} = N^2 \int dp\, e^{-p^2/p_0^2} = N^2\sqrt{p_0^2\pi}$$

with solution $N = (p_0^2\pi)^{-1/4}$. A key step was shifting the integration variable by $\bar{p}$.

(b)

$$\langle p^n\rangle = (p_0^2\pi)^{-1/2}\int_{-\infty}^{\infty} dp\, p^n e^{-(p-\bar{p})^2/p_0^2} = (p_0^2\pi)^{-1/2}\int_{-\infty}^{\infty} dp\,(p+\bar{p})^n e^{-p^2/p_0^2}\ .$$

Therefore

$$\langle p\rangle = (p_0^2\pi)^{-1/2}\int_{-\infty}^{\infty} dp\,(p+\bar{p})e^{-p^2/p_0^2} = \bar{p} \tag{1}$$

since the first term vanishes (p is odd) and the second term is $\bar{p}$ times the normalized integral. We also have

$$\langle p^2\rangle = (p_0^2\pi)^{-1/2}\int_{-\infty}^{\infty} dp\,(p^2+2p\bar{p}+\bar{p}^2)e^{-p^2/p_0^2} = \frac{p_0^2}{2}+0+\bar{p}^2\ , \tag{2}$$

where the first integral (p^2 term) was done in class, the second integral ($2p\bar{p}$ term) vanishes, and the third term ($\bar{p}^2$) is proportional to the normalized integral. Therefore

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{p_0}{\sqrt{2}}\ . \tag{3}$$

(c) In this problem I should have written "$e^{+ipx/\hbar}$" instead of "$e^{-ipx/\hbar}$", but I am solving the problem as written.

$$\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{2\pi\hbar}}(p_0^2\pi)^{-1/4}\int_{-\infty}^{\infty} dp\, e^{-(p-\bar{p})^2/2p_0^2}e^{-ipx/\hbar}\\
&= \frac{1}{\sqrt{2\pi\hbar}}(p_0^2\pi)^{-1/4}\int_{-\infty}^{\infty} dp\, e^{-p^2/2p_0^2}e^{-i(p+\bar{p})x/\hbar}\\
&= \frac{1}{\sqrt{2\pi\hbar}}(p_0^2\pi)^{-1/4}e^{-i\bar{p}x/\hbar}\int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2p_0^2}(p+ip_0^2x/\hbar)^2-p_0^2x^2/(2\hbar^2)}\\
&= \frac{1}{\sqrt{2\pi\hbar}}(p_0^2\pi)^{-1/4}e^{-i\bar{p}x/\hbar}e^{-p_0^2x^2/(2\hbar^2)}\sqrt{2p_0^2\pi}\\
&= (x_0\pi)^{-1/4}\, e^{-x^2/(2x_0^2)}e^{-i\bar{p}x/\hbar}\ ,
\end{aligned} \tag{4}$$

where $x_0 \equiv \hbar/p_0$. You can see by comparison with $\phi(p)$ that this $\psi(x)$ is normalized.

(d) This wave packet represents a particle localized within a region of size $\sim x_0 = \hbar/p_0$ (actually, $\Delta x = x_0/\sqrt{2}$), and which is moving along with average momentum $\bar{p}$ and momentum spread $\Delta p = p_0/\sqrt{2}$.

⋆ **(4)** **(a)** ψ looks like a square wave of height N in the region $-x_0 \le x \le x_0$, vanishing outside this region. Notice that it has infinitely sharp corners. To normalize, $N^2 \int_{-\infty}^{\infty} |\psi|^2 = 1$, or $N = 1/\sqrt{2x_0}$.

(b)

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} = \frac{N}{\sqrt{2\pi\hbar}} \int_{-x_0}^{x_0} e^{-ipx/\hbar} = \frac{\sin(x_0 p/\hbar)}{\sqrt{\pi x_0 p^2/\hbar}} \tag{5}$$

(c)

$$\begin{aligned} \int_{-\infty}^{\infty} dp\, |\phi(p)|^2 &= \left(\frac{1}{\pi x_0/\hbar}\right) \int_{-\infty}^{\infty} dp\, \frac{\sin^2(x_0 p/\hbar)}{p^2} \\ &= \left(\frac{1}{\pi x_0/\hbar}\right) \frac{x_0}{\hbar} \int_{-\infty}^{\infty} d\xi\, \frac{\sin^2 \xi}{\xi^2} = \left(\frac{1}{\pi x_0/\hbar}\right) \frac{x_0}{\hbar} \pi = 1 \ . \end{aligned} \tag{6}$$

Here I substituted integration variables $\xi \equiv x_0 p/\hbar$, and I used the integral for $\int d\xi\, sin^2 \xi/\xi^2$ given in the problem set.

(d)

$$\langle x^n \rangle = \frac{1}{2x_0} \int_{-x_0}^{x_0} dx\, x^n = \frac{x_0^n}{1+n} \left(1 - (-1)^n\right) \ . \tag{7}$$

It follows that $\langle x \rangle = 0$, $\langle x^2 \rangle = x_0^2/3$, $\Delta x = x_0/\sqrt{3}$.

(e)

$$\begin{aligned} \langle p^n \rangle &= \int_{-\infty}^{\infty} dp\, p^n |\phi(p)|^2 = \frac{\hbar}{\pi x_0} \int_{-\infty}^{\infty} dp\, p^{n-2} \sin^2(x_0 p/\hbar) \\ &= \frac{1}{\pi} (\hbar/x_0)^n \int_{-\infty}^{\infty} d\xi\, \xi^{n-2} \sin^2 \xi \ . \end{aligned} \tag{8}$$

It follows that $\langle p \rangle = 0$. However, $\langle p^2 \rangle = \infty$. In order to make the wave function $\psi(x)$ have very sharp edges, we had to use a lot of high frequency components, and so $\phi(p)$ falls off fast enough with respect to p to be normalizable, but not fast enough to give a finite answer for $\langle p^2 \rangle$. Therefore $\Delta p = \infty$.

(f) Evidently, Δx is finite and nonzero for this wavefunction, so $\Delta x \Delta p = \infty$, which is certainly larger than $\hbar/2$, and so is consistent with the uncertainty relation.