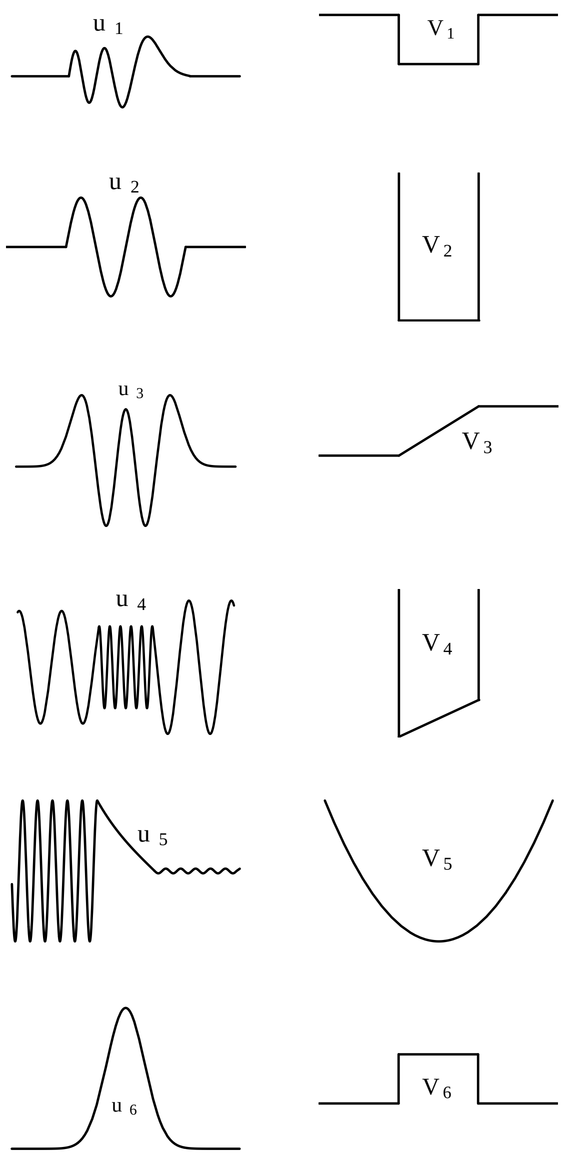
Problem 2. (6 points) In this problem, for each wave function $u_1(x)$, $u_2(x)$,...,$u_6(x)$ which is a solution to a time independent, 1-dimensional Schrödinger equation, you are to write down the corresponding potential (V_1, V_2,..., V_6) that gave rise to that wavefunction. *Note: the subscript on the $u(x)$ does not denote the energy level in this problem.* Plotted are the real parts of the $u(x)$ wave functions. Write your answers in the boxes provided below. Do not assume a one-to-one correspondence between the wave functions and potentials...not all the V's need appear in your answer, and some V's can appear more than once.



wavefunction	u_1	u_2	u_3	u_4	u_5	u_6
potential	$\mathbf{V_4}$	$\mathbf{V_2}$	$\mathbf{V_5}$	$\mathbf{V_1}$	$\mathbf{V_6}$	$\mathbf{V_5}$

(1)

For u_6, I gave full credit for the incorrect answer V_1, since it may be hard to distinguish the differences by eye. One point per answer.

Problem 3. (13 points)

A particle of mass m in a 1 dimensional simple harmonic oscillator potential $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$ (where $\omega \equiv \sqrt{k/m}$) has the initial wavefunction

$$|\psi, t=0\rangle = N\left(\sqrt{2}|0\rangle - |3\rangle\right)$$

where $|n\rangle$, $n = 0, 1, \ldots$, are the harmonic oscillator energy eigenstates, satisfying $\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$, where $\hat{a}^\dagger$ and $\hat{a}$ are the raising and lowering operators respectively. In terms of $\hat{a}$ and $\hat{a}^\dagger$, we have found that the Hamiltonian may be written as $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$

3 a. Find N such that $|\psi, t=0\rangle$ is properly normalized.

$$\begin{aligned} 1 = \langle\psi|\psi\rangle &= |N|^2\left(2\langle 0|0\rangle - \sqrt{2}\langle 0|3\rangle - \sqrt{2}\langle 3|0\rangle + \langle 3|3\rangle\right) \\ &= |N|^2(2+0+0+1) = 3|N|^2 \end{aligned}$$

So the correct answer is $N = \frac{1}{\sqrt{3}}$.

3 b. What is the wavefunction at later time, $|\psi, t\rangle$?

$$\begin{aligned} |\psi, t\rangle &= N\left(\sqrt{2}e^{-iE_0t/\hbar}|0\rangle - e^{-iE_3t/\hbar}|3\rangle\right) \\ &= \frac{1}{\sqrt{3}}\left(\sqrt{2}e^{-i\omega t/2}|0\rangle - e^{-i7\omega t/2}|3\rangle\right) \end{aligned}$$

since $E_n = \hbar\omega(n + 1/2)$.

3 c. Compute the expectation value of the energy $\langle\psi, t|\hat{H}|\psi, t\rangle$ in terms of ω.

There are no cross terms in this matrix element, and so there is no time dependence. (The expectation value of the energy is constant, by energy conservation). So it is simplest to evaluate at $t = 0$.

$$\begin{aligned} \langle\psi, t|\hat{H}|\psi, t\rangle &= |N|^2\left[(\sqrt{2})^2E_0 + (-1)^2E_3\right] = \frac{2}{3}E_0 + \frac{1}{3}E_3 \\ &= \left[\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{7}{2}\right)\right]\hbar\omega = \frac{3}{2}\hbar\omega\ . \end{aligned}$$

Problem 3 continues on the next page $\Longrightarrow$

3 d. If you make a measurement of the particle's energy, what possible values could you measure?

The only possible values you could measure for the energy are $E_0 = \frac{1}{2}\hbar\omega$ or $E_3 = \frac{7}{2}\hbar\omega$.

3 e. What is the most probable result for a measurement of the particle's energy?

You can read off from the wave function at $t = 0$ (with $N = 1/\sqrt{3}$) that there is a $\frac{2}{3}$ probability that it is in the $n = 0$ state, and a $\frac{1}{3}$ probability that it is in the $n = 3$ state. So it is most probable that you will measure $E_0 = \frac{1}{2}\hbar\omega$ for the energy.

3 f. What is the expectation value for the particle's potential energy, $\langle\psi, t|\frac{1}{2}m\omega^2\hat{x}^2|\psi, t\rangle$? *Hint:* Use the expression derived in class, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$.

$$\begin{aligned}
\frac{1}{2}m\omega^2\hat{x}^2 &= \frac{1}{2}m\omega^2\left[\sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\right]^2 \\
&= \frac{\hbar\omega}{4}\left(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}\right) \\
&= \frac{\hbar\omega}{4}\left(\hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1 + \hat{a}^{\dagger 2}\right)
\end{aligned}$$

where I used $[\hat{a}, \hat{a}^\dagger] = 1$. The $\hat{a}^2$ and $\hat{a}^{\dagger 2}$ operators change a state $|n\rangle$ to $|n \pm 2\rangle$, so $\langle 0|\hat{a}^2|0\rangle = \langle 0|\hat{a}^2|3\rangle = 0$, etc. So

$$\begin{aligned}
\langle\psi, t|\frac{1}{2}m\omega^2\hat{x}^2|\psi, t\rangle &= \frac{\hbar\omega}{4}|N|^2\left(2\langle 0|(2\hat{a}^\dagger\hat{a} + 1)|0\rangle + \langle 3|(2\hat{a}^\dagger\hat{a} + 1)|3\rangle\right) \\
&= \frac{\hbar\omega}{12}\left(2(0 + 1) + (6 + 1)\right) = \frac{3}{4}\hbar\omega.
\end{aligned}$$

Problem 3 continues on the next page $\Longrightarrow$

3 g. Suppose you could measure the particle's potential energy with perfect accuracy, and that such a measurement yields the value $\hbar\omega/4$. What is then the expectation value for a subsequent measurement of the particle's kinetic energy? (Careful!)

If you knew the potential energy with compleate accuracy, you would know the value of $\hat{x}^2$ to arbitrary accuracy. Suppose you measured $\langle \frac{1}{2}m\omega^2\hat{x}^2\rangle = \frac{1}{2}m\omega^2\hat{x}_0^2$, where x_0 is some number. Then the wave function would be spiked at $x = \pm x_0$ immediately after the measurement. Thus by the uncertainty principle, the momentum could be anything, and so the expectation value of the kinetic energy would be **infinite**.

What about energy conservation? To measure the potential energy (and hence, the particle's position) to arbitrary accuracy, you needed to shine on it photons with arbitrarily short wavelength, and hence arbitrarily high energy...you blasted the system to pieces with infinite energy photons, so of course the kinetic energy was infinite afterwards.

End of exam