Problem 1. (7 points)
In this problem, consider a particle confined to the region $0 \le x \le L$ with the wavefunction

$$\psi(x,t) = N \sin\left(\frac{2\pi x}{L}\right) e^{-iEt}\,, \qquad E = \frac{2\hbar^2}{mL^2}\,.$$

1 a. Find a value for N that normalizes $\psi(x,t)$.

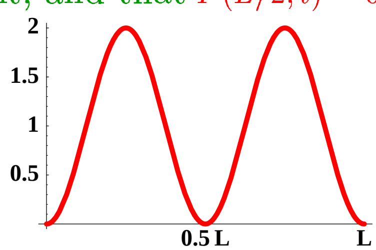
$$\begin{aligned} 1 &= \int_0^L |\psi(x,t)|^2\,dx = |N|^2 \int_0^L \sin^2\left(\frac{2\pi x}{L}\right) dx \\ &= |N|^2 \left(\frac{L}{\pi}\right)^2 \int_0^\pi \sin^2(2y)\,dy \\ &= |N|^2 \left(\frac{L}{\pi}\right)^2 \frac{\pi^2}{2} = |N|^2 \left(\frac{L}{2}\right) \end{aligned} \qquad (1)$$

where I substituted variables $y = \pi x/L$ and used the first integral given on the front page of the exam. So we need $N = \sqrt{2/L}$ (times an arbitrary phase, which we can take to be one).

$$N = \sqrt{2/L}$$

1 b. Sketch the probability density $P(x,t)$ for finding the particle at point x. Label your axes. What is the probability density $P(L/2,t)$ for the particle to be at $x = L/2$ at time t?

$P(x,t) = |\psi(x,t)|^2 = |N|^2 \sin^2(2\pi x/L) = (2/L)\sin^2(2\pi x/L)$. Note that $P(x,t)$ is time independent, and that $P(L/2,t) = 0$.



Problem 1 continued on next page ⟹

1 c. Compute the expectation value for the position, $\langle x\rangle$. (You can express your answer in terms of N, so that if you got part (a) wrong, you can still get this right.)

As should be evident from the sketch of $P(x)$, the probability density is symmetric about $x = L/2$, and so $\langle x\rangle = L/2$. In equations:

$$\begin{aligned}
\langle x\rangle &= \int_0^L x\, P(x)\, dx \\
&= |N|^2 \int_0^L x \sin^2(2\pi x/L)\, dx \\
&= |N|^2 \left(\frac{L}{\pi}\right)^2 \int_0^\pi y \sin^2(2y)\, dy \\
&= |N|^2 \left(\frac{L}{\pi}\right)^2 \frac{\pi^2}{4} = |N|^2 \left(\frac{L}{2}\right)^2 = \frac{L}{2}\,,
\end{aligned} \tag{2}$$

where I substituted variables $y = \pi x/L$ and used the second integral given on the front page of the exam.

$$\boxed{\langle x\rangle = |N|^2 \left(\tfrac{L}{2}\right)^2 = \tfrac{L}{2}}$$

1 d. Compute the expectation value of the square of the momentum, $\langle p^2\rangle$ (again, you needn't substitute in the value for N that you found in (a)).

$$\begin{aligned}
\langle p^2\rangle &= \int_0^L \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi\, dx \\
&= -\hbar^2 |N|^2 \int_0^L \sin(2\pi x/L) \frac{d^2}{dx^2} \sin(2\pi x/L)\, dx \\
&= \hbar^2 \left(\frac{2\pi}{L}\right)^2 |N|^2 \int_0^L \sin^2(2\pi x/L)\, dx \\
&= \hbar^2 \left(\frac{2\pi}{L}\right)^2 .
\end{aligned} \tag{3}$$

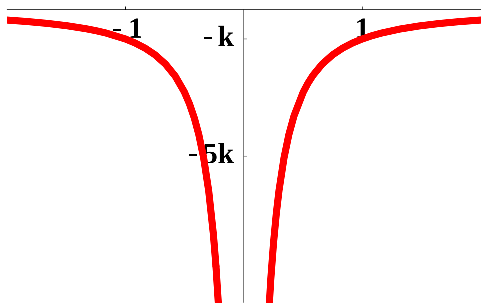
where I took advantage of the fact that the last integral was the same as the normalization integral in (a).

$$\boxed{\langle p^2\rangle = \hbar^2 \left(\tfrac{2\pi}{L}\right)^2 |N|^2 \tfrac{L}{2} = \hbar^2 \left(\tfrac{2\pi}{L}\right)^2}$$

Problem 2 begins on next page $\Longrightarrow$

Problem 2. **(7 points)** Consider a particle of mass m in one dimension, moving in a potential $V(x) = -k/x^{3/2}$, where $k > 0$.

2 a. Sketch $V(x)$ versus x.



2 b. Use the uncertainty principle in the form $\Delta x \Delta p \sim \hbar$ to estimate the ground state energy E_0 for the particle in terms of k, m and $\hbar$.

We take $p \sim \hbar/x$, so that $E = \frac{p^2}{2m} - \frac{k}{|x|^{3/2}} \sim \frac{\hbar^2}{2mx^2} - \frac{k}{|x|^{3/2}}$. Minimizing with respect to x we get

$$0 = \frac{dE}{dx} \sim -\frac{\hbar^2}{mx^3} + \frac{3k}{2|x|^{5/2}} \quad \Longrightarrow \quad \sqrt{x} \sim \frac{2\hbar^2}{3mk} \tag{4}$$

Plugging this value of x back into E we get an estimate for the groundstate energy,

$$E_0 \sim \frac{\hbar^2}{2m}\left(\frac{3mk}{2\hbar^2}\right)^4 - k\left(\frac{3mk}{2\hbar^2}\right)^3 = \left(\frac{k^4m^3}{\hbar^6}\right)\left(\frac{3^4}{2^5} - \frac{3^3}{2^3}\right) = -\frac{27}{32}\left(\frac{k^4m^3}{\hbar^6}\right) . \tag{5}$$

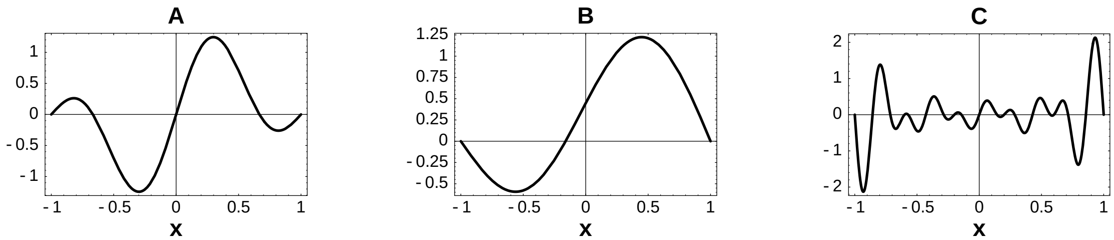
$$E_0 \sim -\frac{27}{32}\left(\frac{k^4m^3}{\hbar^6}\right)$$

2 c. What is the qualitative behavior of your estimate for E_0 if you were to make $\hbar$ smaller? How do you make sense of your answer in the $\hbar \to 0$ limit?

We find $E_0 \to -\infty$ as $\hbar \to 0$. The limit $\hbar \to 0$ is supposed to give you the classical result, and in classical mechanics there is no minimum energy for a particle in a potential that is unbounded below.

Problem 3 begins on next page $\Longrightarrow$

Problem 3. (6 points) Look at the following normalized (real) wavefunctions, plotted as $\psi(x)$ versus x, for a particle in a box $-1 \leq x \leq 1$:



You should be able to answer the following questions without any calculations:

3 a. Which wavefunction (A, B, or C) will yield the largest value for $\langle x \rangle$? Briefly give your reason.

Note that both A and C wave functions are odd in x: $\psi_{A,C}(x) = -\psi_{A,C}(-x)$. Thus both $|\psi_{A,C}|^2$ are even in x, and $\langle x \rangle_{A,C}$ will vanish in both cases. In contrast, ψ_B is asymmetric, with $|\psi_B(x)|^2$ weighted more heavily for $x > 0$, and so $\langle x \rangle_B > 0$.

ψ_B

3 b. Which wavefunction (A, B, or C) will yield the largest value for $\langle x^2 \rangle$? Briefly give your reason.

Both $|\psi_{A,B}|^2$ have most of their weight around $x \sim \pm.5$, while $|\psi_C|^2$ has most of its weight at $x \sim \pm 1$. Thus ψ_C will yield the higher value for $\langle x^2 \rangle$, since $(\pm 1)^2 > (\pm.5)^2$.

ψ_C

3 c. Which wavefunction (A, B, or C) will correspond to the highest kinetic energy? Briefly give your reason.

Kinetic energy $\hat{p}^2/2m$ measures curvature as $\hat{p}^2/2m = -(\hbar^2/2m)d^2/dx^2$, so the wiggliest wavefunction (shortest wavelength) corresponds to the highest momentum, and hence the highest energy. ψ_C wins the prize.

ψ_C

Problem 4 begins on next page $\Longrightarrow$

Problem 4. (5 points) Which is *shorter*: the wavelength of a photon with energy $E = 1$ eV, or the de Broglie wavelength of an electron with kinetic energy $E = 1$ eV? Justify your answer with some equations.

For both electrons and photons we have $\lambda = h/p$, where p is the momentum.

Photons: $p_\gamma = E_\gamma / c \quad \Longrightarrow \quad \lambda_\gamma = hc/E_\gamma$.

Electrons: $p_e = \sqrt{2 m_e E_e} = (\sqrt{2 m_e c^2 E_e})/c \quad \Longrightarrow \quad \lambda_e = hc/\sqrt{2 m_e c^2 E_e}$.

Therefore, with $E_\gamma = E_e \equiv E = 1$ eV, it follows that

$$\frac{\lambda_e}{\lambda_\gamma} = \frac{E_\gamma}{\sqrt{2 m_e c^2 E_e}} = \sqrt{\frac{E}{2 m_e c^2}} \simeq \sqrt{\frac{1 \text{ eV}}{1 \times 10^6 \text{ eV}}} = 1 \times 10^{-3} \ . \tag{6}$$

So the electron's wavelength is much shorter than the photon's, even though they have the same energy. That is why electron microscopes can resolve much smaller objects than optical microscopes.