Problem 2. (10 points)

Consider two **orthonormal** energy eigenstates $|1\rangle$ and $|2\rangle$, where $\hat{H}|1\rangle = E_1|1\rangle$ and $\hat{H}|2\rangle = E_2|2\rangle$, $\hat{H}$ being the Hamiltonian and $E_1 \neq E_2$. Let $|A\rangle$ and $|B\rangle$ define two different linear combinations of the states $|1\rangle$ and $|2\rangle$:

$$|A\rangle \equiv \frac{|1\rangle + i|2\rangle}{\sqrt{2}} , \qquad |B\rangle \equiv \frac{|1\rangle - i|2\rangle}{\sqrt{2}} .$$

2 a. Compute $\langle A|A\rangle$, $\langle B|B\rangle$, $\langle A|B\rangle$ and $\langle B|A\rangle$.

$$\langle A|A\rangle = \langle B|B\rangle = 1 , \qquad \langle A|B\rangle = \langle B|A\rangle = 0 .$$

2 b. If initially at time $t = 0$ the particle is in the state $|\psi, 0\rangle = |A\rangle$, what is the wavefunction $|\psi, t\rangle$ at later times?

$$|\psi, t\rangle = \frac{e^{-iE_1 t/\hbar}|1\rangle + ie^{-iE_2 t/\hbar}|2\rangle}{\sqrt{2}} .$$

2 c.

Suppose you have a measuring device which can tell you whether the particle is in the state $|A\rangle$ or the state $|B\rangle$. For the above initial condition $|\psi, 0\rangle = |A\rangle$, what are the probabilities $P_A(t)$ and $P_B(t)$ that a measurement at time $t > 0$ will find the particle in state $|A\rangle$ or in state $|B\rangle$ respectively? Sketch $P_A(t)$ and $P_B(t)$ as functions of t on the same graph. In the sketch, identify the coordinates $\{t, P(t)\}$ of maxima and minima of the two functions $P_A(t)$ and $P_B(t)$.

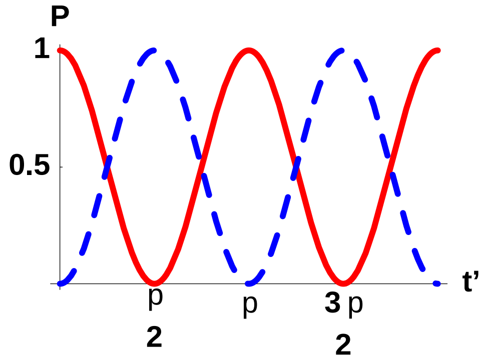
$$\begin{aligned} P_A(t) = |\langle A|\psi, t\rangle|^2 &= \left|\frac{1}{2}\left(e^{-iE_1 t/\hbar} + e^{-iE_2 t/\hbar}\right)\right|^2 \\ &= \left|\frac{1}{2}e^{-i(E_1+E_2)t/2\hbar}\left(e^{-i(E_1-E_2)t/2\hbar} + e^{+i(E_1-E_2)t/2\hbar}\right)\right|^2 \end{aligned}$$

$$= \cos^2[(E_1 - E_2)t/2\hbar] \ . \qquad (1)$$

Similarly,

$$P_B(t) = |\langle B|\psi, t\rangle|^2 = \sin^2[(E_1 - E_2)t/2\hbar] \ .$$

Note that $P_A + P_B = 1$. A plot looks like:



where the horizontal axis is in terms of $t' \equiv t(E_1 - E_2)/2\hbar$.

Problem 3. (10 points)

Consider particles of mass m moving in 1-dimension in the presence of a δ-function potential $V(x) = g\delta(x)$. The time independent Schrödinger equation reads

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + g\delta(x)\right] u(x) = E\, u(x) \ .$$

A beam of monoenergetic particles is incident from the left (negative x) with wavenumber $k = \sqrt{2mE/\hbar^2}$.

- Compute the **transmission probability** for these particles, as a function of k. **Circle your final answer.**
- Does your answer depend on whether the potential is attractive ($g < 0$) or repulsive ($g > 0$)?
- What happens to the transmission probability as $k \to \infty$?

Defining $x < 0$ to be region I and $x > 0$ to region II, we have

$$u_I(x) = Ie^{ikx} + Re^{-ikx} \ , \qquad u_{II}(x) = Te^{ikx} \ , \qquad k = \sqrt{2mE/\hbar^2} \ .$$

I have discarded a solution incoming from the right in region II. The boundary conditions are:

$$u_I(0) = u_{II}(0) \quad \Longrightarrow \quad (I + R) = T \ ;$$

or $R = (T - I)$; for the derivatives, as discussed in class, the δ-function potential gives rise to

$$-\frac{\hbar^2}{2m}\left(u'_{II}(0) - u'_I(0)\right) + g u_{II}(0) = 0$$

or:

$$-\frac{\hbar^2}{2m} ik(T - I + R) + gT = 0 \quad \Longrightarrow \quad T = I\left(\frac{ik\hbar^2}{ik\hbar^2 - mg}\right) .$$

Therefore the transmission probability P_T is

$$P_T = \left|\frac{T}{I}\right|^2 = \frac{k^2\hbar^4}{k^2\hbar^4 + g^2m^2} .$$

Evidently the transmission probability does not depend on the sign of g, and $P_T \to 1$ as $k \to \infty$.

Problem 4. For which particle does the wavefunction have the **largest** magnitude for the frequency ν?

(a) A photon with momentum $p_\gamma = 1\ \text{eV}/c$ (the photon mass is $m_\gamma = 0$).

(b) An electron with momentum $p_e = 1\ \text{eV}/c$ (the electron mass is $m_e = 0.5\ \text{MeV}/c^2$).

(c) An proton with momentum $p_p = 1\ \text{eV}/c$ (the proton mass is $m_p = 938\ \text{MeV}/c^2$).

(d) A neutron with momentum $p_n = 1\ \text{eV}/c$ (the neutron mass is $m_n = 940\,,\ \text{MeV}/c^2$).

(e) An electron in the ground state of the hydrogen atom.

The answer is **(e)**. From Planck and de Broglie we know that $E = h\nu$, so the larger the energy, the larger ν. We have $E_\gamma = cp_\gamma = 1\ \text{eV}$. For the electron, proton, and neutron in (b-d), $E = p^2/(2m) \ll 1\ \text{eV}$. The binding energy of the H atom electron is $13.6\ \text{eV}$ and wins the prize.

Problem 5. Which two quantities describing a particle's motion in a central potential in 3-dimensions **can not** be simultaneously measured with arbitrary accuracy, even in principle:

(a) y position and the z-component of angular momentum.

(b) Total angular momentum and the x-component of angular momentum.

(c) x position and the x-component of angular momentum.

(d) Energy and total angular momentum.

(e) Energy and the x-component of angular momentum.

The answer is **(a)**. Since $\vec{L} = \vec{r} \times \vec{p}$, $p_z = (xp_y - yp_x)$, and y doesn't commute with p_y. Compare with (c) where $L_x = (yp_z - zp_y)$, all terms of which commute with x.

Problem 6. Suppose I have a particle of mass m in some quantum state $|\psi, t\rangle$ of a 1-dimensional harmonic oscillator with potential $V(x) = \frac{1}{2}kx^2$. The corresponding classical frequency $\omega = \sqrt{k/m}$. Which one equation below is **false**?

(a) $\frac{d}{dt}\langle\psi, t|\hat{x}|\psi, t\rangle = \langle\psi, t|\hat{p}|\psi, t\rangle/m$

(b) $\frac{d}{dt}\langle\psi, t|\hat{p}|\psi, t\rangle = -k\langle\psi, t|\hat{x}|\psi, t\rangle$

(c) $\langle\psi, t|\hat{x}|\psi, t\rangle = x_0 \cos(2\omega t)$, for some nonzero constant x_0

(d) $\langle\psi, t|\hat{x}^2|\psi, t\rangle = x_0^2$, for some nonzero constant x_0

(e) $\langle\psi, t|\hat{H}|\psi, t\rangle = 2\hbar\omega$

The answer is **(c)**. You showed that (a) and (b) were generally true in a problem set. Putting them together yields $d^2\langle x\rangle/dt^2 = -\omega^2\langle x\rangle$, the classical equation for the position...the general solution is $\langle x\rangle = x_0\cos(\omega t + \delta)$, and a frequency of 2ω is impossible. Also easy to see using raising and lowering operators. As for (d) and (e): we can have $\langle x^2\rangle = x_0^2$ for some x_0 (for example, in the ground state), and the expectation value of the energy can equal $2\hbar\omega$ even if there is no energy eigenstate with that energy.

Problem 7. Suppose at $t = 0$ a particle is in the state

$$|\psi, 0\rangle = N\left(|1\rangle - i|2\rangle + 2|3\rangle + \sqrt{3}|4\rangle\right)$$

where $|n\rangle$ are the orthonormal eigenstates of some Hamiltonian with $\hat{H}|n\rangle = E_n|n\rangle$. The number N is chosen to normalize the state so that $\langle\psi, 0|\psi, 0\rangle = 1$. Which of the following statements is <u>false</u>?

(a) At later times t, $\langle\psi, t|\psi, t\rangle = 1$

(b) The normalization constant can be taken to be $N = 1/3$.

(c) The expectation value of the Hamiltonian, $\langle\psi, t|\hat{H}|\psi, t\rangle$, is time independent.

(d) The probability of measuring the energy to be E_3 is twice the probability of measuring the energy to be E_1.

(e) A measurement of the particle's energy can only yield the values E_1, E_2, E_3 or E_4.

The answer is **(d)**, since the probability for measuring E_3 is $2^2 = 4$ times more likely than measuring E_1. All the other statements are true.

Problem 8. The wavefunction of a diatomic molecule is described by $\psi(\theta, \phi)$, where θ and ϕ are the polar and azimuthal angles respectively describing the orientation of the axis connecting the two atoms. Suppose that

$$\psi(\theta, \phi) = N \sin^2\theta \, e^{i\phi} \ .$$

Which of one of the following combinations of values for ℓ and m **could** result from a measurement of L^2 and L_z for this state?

(a) $\ell = 0$ and $m = 0$

(b) $\ell = 1$ and $m = 0$

(c) $\ell = 2$ and $m = 0$

(d) $\ell = 1$ and $m = 1$

(e) $\ell = 2$ and $m = 1$

(f) $\ell = 2$ and $m = 2$

(g) None of the above

For this problem, here is a list of relevant spherical harmonics $Y_{\ell,m}(\theta, \phi) = \langle \theta, \phi | \ell, m \rangle$*:*

$$\begin{aligned} Y_{0,0} &= \sqrt{\tfrac{1}{4\pi}} & Y_{1,0} &= \sqrt{\tfrac{3}{4\pi}} \cos\theta & Y_{2,0} &= \sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta - 1) \\ & & Y_{1,1} &= -\sqrt{\tfrac{3}{8\pi}} e^{i\phi} \sin\theta & Y_{2,1} &= -\sqrt{\tfrac{15}{8\pi}} e^{i\phi} \sin\theta\cos\theta \\ & & & & Y_{2,2} &= \sqrt{\tfrac{15}{32\pi}} e^{2i\phi} \sin^2\theta \end{aligned}$$

The answer is **(d)**. *The probability for measuring a particular* ℓ, m *pair is* $P_{\ell,m} = |\langle \ell, m|\psi\rangle|^2$, *where*

$$\langle \ell, m|\psi\rangle = \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \sin\theta \, \langle \ell, m|\theta, \phi\rangle\langle\theta, \phi|\psi\rangle = \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \sin\theta \, Y^*_{\ell,m}(\theta, \phi)\psi(\theta, \phi) \ .$$

In this particular case, to avoid having the ϕ integral vanish, we must have $Y_{\ell,m} \propto e^{i\phi}$; that leaves only (d), (e) or (g) as possibilities. THen the integration over ϕ yields 2π, and we must look at the integration of θ. For $\ell = m = 1$,

$$P_{1,1} = -2\pi N \sqrt{\frac{3}{8\pi}} \int_0^{\pi} d\theta \sin^4 \theta$$

which is clearly nonzero, as the integrand is positive everywhere. On the other hand

$$P_{2,1} = -2\pi N \sqrt{\frac{15}{8\pi}} \int_0^{\pi} d\theta \sin^3 \theta \cos \theta$$

which vanishes as $\sin\theta$ is even and $\cos\theta$ is odd over the interval $\theta = [0, \pi]$.

End of exam...have a good vacation

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	